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STRENGTHENING PARAMETRIZED-DIFFERENCE BELIEF REVISION

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Abstract. Parametrized-difference (PD) belief revision is a special type of rational belief change, based on a fixed ranking over the atoms of the underlying language, with a plethora of appealing characteristics. The process of PD revision is encoded in the so-called PD operators, which essentially constitute a particular class of rational revision functions. In this article, we strengthen the PD-revision framework with respect to three aspects that are not present in its original proposal. In particular, we first define a dynamic form of PD revision, letting a changeable ranking over atoms. Furthermore, we show that PD operators are incompatible with Horn revision. Against this background, alternative Horn compliant revision operators, in the spirit of PD operators, are introduced. Lastly, we study PD revision in the realm of Description Logics, a family of knowledge representation languages oriented towards practical applications.

1. Introduction

Belief Revision studies the dynamics of knowledge [14, 12]. The *AGM paradigm*, named after its original developers Alchourrón, Gärdenfors and Makinson [1], is the widely-accepted framework modelling the belief-change process. Within this framework, an agent's belief corpus is represented as a logical theory, the new information (*epistemic input*) is represented as a logical sentence, and the policy of belief revision is encoded in a *revision function* that maps a theory and a sentence to a revised (new) theory. *Rational* revision functions, called *AGM revision functions*, are characterized *axiomatically* by a set of eight postulates, known as the *AGM postulates for revision*, and *constructively*, by means of a special kind of total preorders over possible worlds, called *faithful preorders* [18].

A well-behaved family of concrete AGM revision functions, called *Parametrized-Difference revision operators* (alias, *PD operators*), has been recently introduced by Peppas and Williams [26, 27]. PD operators are natural generalizations of the *Hamming-based* Dalal's revision operator [9], and are expressive enough to cover a variety of real-world applications.

The construction of a PD operator requires an agent to specify a single total preorder $\trianglelefteq$ over all propositional variables (atoms) of the underlying language, which essentially encodes their (prior) relative epistemic value. This *low representational cost* is an important feature of PD

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revision, given that the construction of an arbitrary AGM revision function requires a total preorder over *all* possible worlds, for *every* belief set of the language.[1]

Perhaps more importantly, as far as practical applications are concerned, when confined to Horn knowledge bases and the size of queries is bounded by a constant, the complexity of PD operators drops to *linear time* with respect to the size of the knowledge base [27].[2]

The aim of the present article is to strengthen the existing PD-revision framework, by studying the following three aspects that are not present in the original proposal of Peppas and Williams. Specifically:

- We consider the case where the epistemic input causes *changes* in the preorder $\trianglelefteq$ over atoms. The proposed method demands *no extra representational cost*, thus, the benefits of PD operators concerning compact specification are preserved. Since $\trianglelefteq$ *uniquely* defines a single PD operator, its variation results in changes to the underlying revision policy, a fact that in turn implies a form of *dynamic* PD revision.
- As *Horn logic* (namely, the Horn fragment of propositional logic) has been applied numerous times in both Artificial Intelligence and databases, we examine whether PD operators are compatible with *Horn revision*; i.e., whether the PD-revision of a Horn knowledge base by a Horn formula *always* yields a (new) Horn knowledge base. An established result shows that, unfortunately, this is not the case.[3]

  Against this background, an alternative way for defining Horn compliant revision operators, parametrized by a preorder over atoms, is proposed. Furthermore, on top of very recent results established by the author (along with Peppas and Williams) [5], an indirect connection between PD and Horn revision, by an *extension* of the underlying language, is specified.
- Although the study of belief revision is confined, mainly, to the propositional setting, several important attempts have been made for generalizing the classical AGM paradigm to modern logical formalisms, such as *Description Logics* (DL) [13, 31, 29]. DL are decidable fragments of first-order logic, widely used in ontological modelling and the Semantic Web, and, unlike a propositional language that is built from atoms, DL are built from atomic *concepts* (unary predicates) and atomic *roles* (binary predicates) [7].

  Following this line of research, a study on the adaptation of PD revision in the realm of DL is conducted herein. In particular, *Hamming-based* DL revision operators are introduced, as well as various modified forms of them, *parametrized* by a total preorder over all concept and role names.

The remainder of this article is structured as follows: The next section provides the basic formal preliminaries, followed by a brief summary of the AGM paradigm (Section 3). Sections 4 and 5 introduce Dalal's construction and PD operators, respectively. Section 6 presents dynamic PD revision, Section 7 studies the relation between PD and Horn revision, and Section 8 examines PD revision in DL. The last section of the paper is devoted to a brief conclusion.

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[1] For a propositional language built over n atoms, there exist 2^n possible worlds and $2^{(2^n)}$ belief sets.

[2] Recall that a Horn knowledge base is a set of Horn clauses, where a Horn clause is a clause (a disjunction of literals) with at most one positive literal.

[3] This does not entail that the PD-revision of a Horn knowledge base by a Horn formula —which, as stated, is characterized by low computational cost— is not a legitimate operation; all that Horn non-compliance entails is that the result of the revision *may be* a non-Horn knowledge base.

2. Formal Preliminaries

This section fixes basic notation and terminology that shall be used throughout the article.

2.1. **Language.** For a *finite*, non-empty set of propositional variables (alias, atoms) $\mathcal{P}$, we define $\mathcal{L}$ to be the propositional language generated from $\mathcal{P}$, using the standard Boolean connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\leftrightarrow$ (equivalence), $\neg$ (negation), the special symbol $\bot$ (arbitrary contradiction), and governed by *classical propositional logic*.

A sentence φ of $\mathcal{L}$ is *contingent* iff $\nvDash \varphi$ and $\nvDash \neg\varphi$. For a set of sentences Γ of $\mathcal{L}$, $Cn(\Gamma)$ denotes the set of sentences following logically from Γ; i.e.,

$$Cn(\Gamma) = \big\{\varphi \in \mathcal{L} : \Gamma \models \varphi\big\}.$$

We shall write $Cn(\varphi_1, \ldots, \varphi_n)$ for sentences $\varphi_1, \ldots, \varphi_n$, as an abbreviation of $Cn\big(\{\varphi_1, \ldots, \varphi_n\}\big)$, and $\varphi \equiv \psi$ iff $Cn(\varphi) = Cn(\psi)$, for any two sentences $\varphi, \psi \in \mathcal{L}$.

2.2. **Belief Sets.** An agent's set of beliefs will be modelled as a *theory*, also referred to as a *belief set*. A theory K of $\mathcal{L}$ is any set of sentences of $\mathcal{L}$ closed under logical consequence; in symbols,

$$K = Cn(K).$$

Note that, since $\mathcal{L}$ is built from a finite set of atoms, any theory K can be represented (modulo logical equivalence) as a single sentence χ; that is, $K = Cn(\chi)$.

We denote the set of all consistent theories of $\mathcal{L}$ by $\mathbb{K}$. A theory K is *complete* iff, for all sentences $\varphi \in \mathcal{L}$, either $\varphi \in K$ or $\neg\varphi \in K$. For a theory K and a sentence φ of $\mathcal{L}$, $K + \varphi$ abbreviates the theory $Cn\big(K \cup \{\varphi\}\big)$.

2.3. **Literals and Possible Worlds.** A *literal* is a propositional variable $p \in \mathcal{P}$ or its negation. We define a *possible world* (or simply a *world*) r to be a consistent set of literals, such that, for any atom $p \in \mathcal{P}$, either $p \in r$ or $\neg p \in r$. The set of all possible worlds is denoted by $\mathbb{M}$.

For a sentence (set of sentences) φ of $\mathcal{L}$, $[\varphi]$ is the set of worlds at which φ is true. For the sake of readability, the negation of a propositional variable p will, sometimes, be represented as $\overline{p}$, instead of $\neg p$. Moreover, possible worlds will, occasionally, be represented as sequences (rather than sets) of literals. For a set of literals Q, we denote by Q^+ the set containing all positive literals (atoms) of Q; i.e.,

$$Q^+ = Q \cap \mathcal{P}.$$

2.4. **Preorders.** A *preorder* over a set V is any reflexive, transitive binary relation in V. A preorder $\preceq$ is called *total* iff, for all $r, r' \in V$, $r \preceq r'$ or $r' \preceq r$. As usual, we shall denote by $\prec$ the strict part of $\preceq$; i.e., $r \prec r'$ iff $r \preceq r'$ and $r' \npreceq r$. Moreover, we shall denote by $\approx$ the symmetric part of $\preceq$; i.e., $r \approx r'$ iff $r \preceq r'$ and $r' \preceq r$.

For any set $X \subseteq V$, by $min(X, \preceq)$ we denote the set of all $\preceq$-minimal elements of X; in symbols,

$$min(X, \preceq) = \Big\{r \in X : \text{for all } r' \in X, \text{ if } r' \preceq r, \text{ then } r \preceq r'\Big\}.$$

Whenever the set X contains (natural) numbers, we shall simply write $min(X)$ to denote the minimum number in X.

3. The AGM Paradigm

In this section, the *postulational* side of the AGM paradigm, as well as the *faithful-preorders model* for the process of belief revision, are briefly presented.

In the course of this work, we shall consider, for ease of presentation, only the principal case of *consistent* belief sets and *contingent* epistemic input, unless explicitly stated otherwise. Note, however, that the AGM paradigm treats the limiting cases of inconsistent belief sets and non-contingent epistemic input as well. In any case, its dictates are that the revised belief set is a consistent theory of $\mathcal{L}$, unless the epistemic input is itself inconsistent; see postulate $(K * 5)$ of the next subsection.

3.1. The AGM Postulates for Revision.

Within the AGM paradigm, the process of belief revision is modelled as a (binary) function $*$, mapping a consistent theory K and a contingent sentence φ to a consistent (revised) theory $K * \varphi$ (Figure 1). The *AGM postulates for revision* $(K * 1)$–$(K * 8)$, presented subsequently, circumscribe the territory of all *rational* revision functions, the so-called *AGM revision functions*.[4]

(K ∗ 1) $K * \varphi$ is a theory of $\mathcal{L}$.

(K ∗ 2) $\varphi \in K * \varphi$.

(K ∗ 3) $K * \varphi \subseteq K + \varphi$.

(K ∗ 4) If $\neg\varphi \notin K$, then $K + \varphi \subseteq K * \varphi$.

(K ∗ 5) $K * \varphi$ is inconsistent iff $\models \neg\varphi$.

(K ∗ 6) If $\varphi \equiv \psi$, then $K * \varphi = K * \psi$.

(K ∗ 7) $K * (\varphi \wedge \psi) \subseteq (K * \varphi) + \psi$.

(K ∗ 8) If $\neg\psi \notin K * \varphi$, then $(K * \varphi) + \psi \subseteq K * (\varphi \wedge \psi)$.

Postulates $(K * 1)$–$(K * 8)$ have been slightly reshaped by Katsuno and Mendelzon for the propositional setting, in order to make the formalization more amendable to implementations [18]. In particular, Katsuno and Mendelzon assume that the beliefs of an agent are represented by a sentence ψ of $\mathcal{L}$, and the result of the revision of ψ by an epistemic input μ of $\mathcal{L}$ is, also, a sentence of $\mathcal{L}$. The resulting postulates, which are equivalent to the AGM postulates for revision $(K * 1)$–$(K * 8)$, are presented below.

(KM1) $\psi * \mu \models \mu$.

(KM2) If $\psi \wedge \mu$ is consistent, then $\psi * \mu \equiv \psi \wedge \mu$.

(KM3) If μ is consistent, then $\psi * \mu$ is also consistent.

(KM4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$, then $\psi_1 * \mu_1 \equiv \psi_2 * \mu_2$.

(KM5) $(\psi * \mu) \wedge \varphi \models \psi * (\mu \wedge \varphi)$.

(KM6) If $(\psi * \mu) \wedge \varphi$ is satisfiable, then $\psi * (\mu \wedge \varphi) \models (\psi * \mu) \wedge \varphi$.

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[4] For a detailed elaboration on the postulates, refer to [14] or [25].

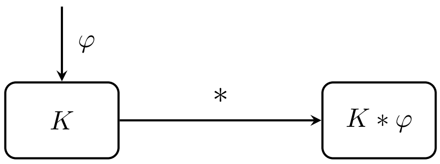

FIGURE 1. Belief revision within the AGM paradigm. (Figure borrowed from [6].)

3.2. **Faithful-Preorders Model.** Apart from the aforementioned *axiomatic* approach, several *constructive models* for the process of belief revision have been proposed. Herein, we discuss the well-known model introduced by Katsuno and Mendelzon, which is based on a special kind of total preorders over all possible worlds, called *faithful preorders* [18].

Definition 1 (Faithful Preorder, [18])**.** For a theory K of $\mathcal{L}$, a preorder over possible worlds $\preceq_K$ is faithful to K iff it is total, and such that, for any two possible worlds $r, r' \in \mathbb{M}$:[5]

- **(i)** If $r \in [K]$, then $r \preceq_K r'$.
- **(ii)** If $r \in [K]$ and $r' \notin [K]$, then $r \prec_K r'$.

The above definition specifies the belief set K as the set corresponding to the $\preceq_K$-minimal worlds; i.e., $[K] = min(\mathbb{M}, \preceq_K)$.

A function that maps *each* theory K of $\mathcal{L}$ to a preorder $\preceq_K$, faithful to K, is called a *faithful assignment*. Based on the notion of faithful assignment, the following representation theorem relates the axiomatic and constructive sides of the belief-revision process.

Theorem 1 ([18])**.** *A revision operator* $*$ *satisfies postulates* $(K*1)$–$(K*8)$ *iff there exists a faithful assignment such that, for every* $K \in \mathbb{K}$ *and* $\varphi \in \mathcal{L}$*:*

$$\textbf{(F}*\textbf{)} \quad [K * \varphi] = min([\varphi], \preceq_K).$$

From an epistemological point of view, the faithful preorder $\preceq_K$ encodes the *comparative plausibility* of the possible worlds of $\mathbb{M}$, relative to K, with the more plausible worlds appearing lower in the ordering. Hence, condition (F$*$) defines the revised belief set $K * \varphi$ as the theory corresponding to the most plausible (with respect to K) φ-worlds.

4. Dalal's Revision Operator

For a theory K of $\mathcal{L}$, Dalal defines the plausibility of possible worlds, encoded in a preorder $\preceq_K$ faithful to K, in terms of a *Hamming-based difference* between worlds [9]. In the limiting case where theory K is inconsistent (i.e., $[K] = \varnothing$), Dalal defines the belief set resulting from the revision of K by φ to be equal to $Cn(\varphi)$. For the principal case of a consistent theory K, he proceeds to the following definitions.

Definition 2 (Difference between Worlds)**.** The difference between two worlds w, r of $\mathbb{M}$, denoted by $\mathit{Diff}\,(w, r)$, is the set of propositional variables over which the two worlds disagree. In symbols:

$$\mathit{Diff}\,(w, r) = \Big((w - r) \cup (r - w)\Big) \cap \mathcal{P}.$$

[5] In [18], faithful preorders are associated with sentences, rather than theories. However, given that the underlying propositional language is built over a finite set of atoms, the difference is immaterial.

Definition 3 (Distance between Theories and Worlds, [9])**.** The distance between a theory K of $\mathcal{L}$ and a world r of $\mathbb{M}$, denoted by $Dist\ (K, r)$, is as follows:

$$Dist\ (K, r) = min\Big(\Big\{\big|Diff\ (w, r)\big| : w \in [K]\Big\}\Big).$$

Definition 4 (Dalal's Operator, [9])**.** Dalal's operator is the revision function induced, via condition (F$*$), from the family of Dalal's preorders $\{\sqsubseteq_K\}_{K \in \mathbb{K}}$, where each preorder $\sqsubseteq_K$ is defined, for any $r, r' \in \mathbb{M}$, by means of condition (D).

$$\textbf{(D)} \quad r \sqsubseteq_K r' \ \text{ iff } \ Dist\ (K, r) \leqslant Dist\ (K, r').$$

It can be easily shown that, for each theory K of $\mathcal{L}$, $\sqsubseteq_K$ is a total preorder faithful to K, therefore, Dalal's operator satisfies the full set of AGM postulates for revision [18, p. 269].

5. Parametrized-Difference Revision Operators

A natural generalization of Dalal's operator has been recently introduced by Peppas and Williams, who defined *axiomatically* in [27] and *constructively* in [26] a new family of concrete AGM revision operators, called *Parametrized-Difference revision operators*; for short, *PD operators*. PD operators are wide enough to cover a plethora of different applications, and have nice computational properties in a Horn setting (provided that the size of queries is bounded by a constant). Furthermore, they can be compactly represented, as each PD operator can be fully specified from a single total preorder $\trianglelefteq$ over atoms (rather than over possible worlds), which essentially encodes their (prior) relative epistemic value; in particular, the more epistemic entrenched (and, thus, more resistant to change) an atom is, the higher it appears in $\trianglelefteq$.

Subsequently, we proceed to the formal definition of $\trianglelefteq$, as presented in [26]. Let $\trianglelefteq$ be a total preorder over the set $\mathcal{P}$ of atoms. For a set of atoms $\mathcal{S}$ and an atom q, by $\mathcal{S}_q$ we denote the set $\mathcal{S}_q = \{p \in \mathcal{S} : p \trianglelefteq q\}$. The preorder $\trianglelefteq$ can, then, be extended to *sets* of propositional variables.

Definition 5 (Preorder over Sets of Atoms, [26])**.** For any two sets of atoms $\mathcal{S}$, $\mathcal{S}'$, $\mathcal{S} \trianglelefteq \mathcal{S}'$ iff one of the following three conditions holds:

- **(i)** $|\mathcal{S}| < |\mathcal{S}'|$.
- **(ii)** $|\mathcal{S}| = |\mathcal{S}'|$, and for all $q \in \mathcal{P}$, $|\mathcal{S}_q| = |\mathcal{S}'_q|$.
- **(iii)** $|\mathcal{S}| = |\mathcal{S}'|$, and for some $q \in \mathcal{P}$, $|\mathcal{S}_q| > |\mathcal{S}'_q|$, and for all $p \triangleleft q$, $|\mathcal{S}_p| = |\mathcal{S}'_p|$.[6]

In the above definition, condition (ii) states that $\mathcal{S}$ and $\mathcal{S}'$ are *lexicographically indistinguishable* (with respect to $\trianglelefteq$), whereas, condition (iii) states that $\mathcal{S}$ *lexicographically proceeds* $\mathcal{S}'$ (with respect to $\trianglelefteq$). It turns out that the extended $\trianglelefteq$ is a total preorder over $2^{\mathcal{P}}$.

The intended reading of the extended preorder $\trianglelefteq$, defined over sets of atoms, is the same as in the case of individual atoms; namely, $\mathcal{S} \trianglelefteq \mathcal{S}'$ asserts that a change of *all* atoms in $\mathcal{S}'$ is less plausible than a change of *all* atoms in $\mathcal{S}$.

Definition 6 (PD Operator, [26])**.** Let $\trianglelefteq$ be a total preorder over the set of propositional variables $\mathcal{P}$. A PD operator is the revision function induced, via condition (F$*$), from the family of PD preorders $\{\sqsubseteq^{\trianglelefteq}_K\}_{K \in \mathbb{K}}$, where each preorder $\sqsubseteq^{\trianglelefteq}_K$ is defined, for any $r, r' \in \mathbb{M}$, by means of condition (PD).

[6] $\triangleleft$ denotes the strict part of $\trianglelefteq$.

$$\textbf{(PD)} \quad r \sqsubseteq_K^{\trianglelefteq} r' \quad \text{iff} \quad \begin{array}{l} \text{there is a } w \in [K], \text{ such that, for all } w' \in [K], \\ \mathit{Diff}\,(w, r) \trianglelefteq \mathit{Diff}\,(w', r'). \end{array}$$

The results in [26] imply that $\sqsubseteq_K^{\trianglelefteq}$ is a total preorder, faithful to K. Observe that, when $\trianglelefteq = \mathcal{P} \times \mathcal{P}$, then the PD preorder $\sqsubseteq_K^{\trianglelefteq}$ reduces to Dalal's preorder $\sqsubseteq_K$.

Definition 6 dictates that a *single* preorder $\trianglelefteq$ generates a *unique family* of PD preorders $\{\sqsubseteq_K^{\trianglelefteq}\}_{K \in \mathbb{K}}$, which in turn defines, via (F$*$), a revision function $*$. A revision function so constructed is called PD operator. The one-to-one correspondence —which can be easily shown from Definition 6, given that $\mathcal{P}$ is finite— between preorders over atoms and PD preorders is illustrated in Figure 2.

Let us, now, examine the relationship between Dalal's construction and PD operators through the following concrete example, borrowed from [26].

Example 1 ([26])**.** Let $\mathcal{P} = \{a, b, c\}$, and let K be the complete theory $K = Cn(a, b, c)$. Then, the (only) preorder that Dalal attaches to K is the following:[7]

$$abc \quad \sqsubset_K \quad \begin{array}{c} \overline{a}bc \\ a\overline{b}c \\ ab\overline{c} \end{array} \quad \sqsubset_K \quad \begin{array}{c} \overline{a}\overline{b}c \\ \overline{a}b\overline{c} \\ a\overline{b}\overline{c} \end{array} \quad \sqsubset_K \quad \overline{a}\overline{b}\overline{c}$$

According to Dalal, the plausibility of a world r is determined by the number of propositional variables on which r differs from the unique world abc of the complete theory K. Dalal's approach considers that all variables have the same epistemic value; hence, for instance, a change in variable a is assumed to be as plausible (or implausible) as a change in variable b.

To see how the aforementioned Dalal's preorder can be restructured to a PD preorder, consider a total preorder $\trianglelefteq$ over $\mathcal{P}$, such that $c \lhd a \sim b$ (i.e., variables a and b have greater epistemic value than variable c, and, consequently, a change in a or b is less plausible than a change in c).[8] Given $\trianglelefteq$, Dalal's preorder is refined as follows:[9]

$$abc \quad \sqsubset_K^{\trianglelefteq} \quad ab\overline{c} \quad \sqsubset_K^{\trianglelefteq} \quad \begin{array}{c} \overline{a}bc \\ a\overline{b}c \end{array} \quad \sqsubset_K^{\trianglelefteq} \quad \begin{array}{c} \overline{a}b\overline{c} \\ a\overline{b}\overline{c} \end{array} \quad \sqsubset_K^{\trianglelefteq} \quad \overline{a}\overline{b}c \quad \sqsubset_K^{\trianglelefteq} \quad \overline{a}\overline{b}\overline{c}$$

The above ranking takes place in two stages. The first stage is identical to Dalal's approach; each world r is ranked according to the number of switches in propositional variables that are necessary to turn the initial world abc into r. In the second stage, the ranking is further refined to take into account the different epistemic values of the atoms (encoded in $\trianglelefteq$) that have been switched.

We close this section by mentioning that the characterization of PD operators in the realm of all popular constructive models for belief revision, as well as their full compliance with Parikh's *relevance-sensitive* axiom [24], were established in [4].[10]

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[7] $\sqsubset_K$ denotes the strict part of $\sqsubseteq_K$.

[8] $\sim$ denotes the symmetric part of $\trianglelefteq$.

[9] $\sqsubset_K^{\trianglelefteq}$ denotes the strict part of $\sqsubseteq_K^{\trianglelefteq}$.

[10] Parikh's axiom is a well-studied constraint that addresses the weakness of the AGM postulates for revision in capturing relevant change.

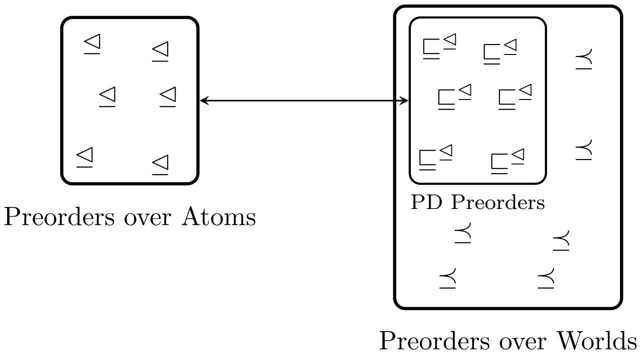

FIGURE 2. The one-to-one correspondence between preorders over atoms and PD preorders. Notation for preorders over worlds represents families of preorders (one preorder for each theory K of $\mathcal{L}$).

6. Dynamic PD Revision

We, now, turn to the first contribution of this article, which is the definition of a natural form of *dynamic* PD revision.

As stated, once the total preorder $\trianglelefteq$ over $\mathcal{P}$ has been chosen by a rational agent, the family of preorders $\{\sqsubseteq_K^{\trianglelefteq}\}_{K \in \mathbb{K}}$ is *uniquely* determined (via condition (PD)). As $\{\sqsubseteq_K^{\trianglelefteq}\}_{K \in \mathbb{K}}$ essentially encodes the revision policy of the agent, a constant preorder $\trianglelefteq$ sets, in turn, a *static* revision policy. Clearly, however, real-world agents do change the way they revise their beliefs at certain applications, as a result of rearranging the relative epistemic value of propositions.[11]

Evidently, the only way to change the revision policy of an agent and, simultaneously, to remain in the realm of PD revision, is to change the preorder $\trianglelefteq$. A plausible way to change $\trianglelefteq$ would be by means of the epistemic input φ; after all, φ is the *only external information* that the agent receives through the whole revision process. And then we come to the natural question: How exactly the formula φ could initiate a change in $\trianglelefteq$? Subsequently, we present our proposal.

Firstly, we assume that not every formula (epistemic input) can cause a change (substitution) of the preorder $\trianglelefteq$, but only some *critical* ones. For determining critical formulas, a formal *substitution policy* is required. To this aim, consider the following definition.

Definition 7 (Degree of Justification of Atoms in Sentences)**.** Let φ be a sentence of $\mathcal{L}$, and let p be a propositional variable of $\mathcal{P}$. We define the function ρ to be a mapping from φ, p to the set of non-negative integers, such that:

$$\rho\ (\varphi, p) = \big|\ [\varphi \wedge p]\ \big|.$$

Essentially, $\rho\ (\varphi, p)$ represents the *number* of φ-worlds that satisfy (entail) the atom p, and, in a sense, expresses a quantitative degree of the *deductive justification* that p has in φ — for analogous qualitative degrees of deductive justification, refer to [2, 3, 6]. This degree of deductive justification will, in turn, serve as a means of the specification of a *new* total preorder $\trianglelefteq^{\varphi}$ over $\mathcal{P}$, defined as follows:

[11]From an epistemological perspective, such changes are strongly related to what Kuhn calls *paradigm shifts*, in the realm of scientific theories [20] — refer to the discussion of Section 4.7 in [1, pp. 91–94].

$$a \trianglelefteq^{\varphi} b \quad \text{iff} \quad \rho\ (\varphi, a) \leqslant \rho\ (\varphi, b), \quad \text{for any } a, b \in \mathcal{P}.$$

Clearly, each epistemic input φ induces, via ρ, a total preorder $\trianglelefteq^{\varphi}$ over atoms, which encodes a *new* relative epistemic value that the agent assigns to the atoms of the language. Specifically, according to $\trianglelefteq^{\varphi}$, an atom b is epistemically more entrenched than an atom a whenever b is entailed/justified by more φ-worlds than a.

Example 2. Assume that $\mathcal{P} = \{a, b, c\}$, and $\varphi = (a \vee \neg b) \wedge c$. Clearly then, $[\varphi] = \{abc, a\bar{b}c, \bar{a}\bar{b}c\}$, and $\rho\ (\varphi, a) = 2$, $\rho\ (\varphi, b) = 1$, $\rho\ (\varphi, c) = 3$. Therefore, we have the following preorder $\trianglelefteq^{\varphi}$ over $\mathcal{P}$: $b \triangleleft^{\varphi} a \triangleleft^{\varphi} c$.

A natural candidate for comparing the *similarity* (or *dissimilarity*) between the preorders $\trianglelefteq$ and $\trianglelefteq^{\varphi}$ would be the *Kemeny distance* [19], defined subsequently.

Definition 8 (Kemeny Distance, [19])**.** Let $\trianglelefteq$, $\trianglelefteq'$ be two total preorders over the atoms of $\mathcal{P}$. The Kemeny distance between $\trianglelefteq$, $\trianglelefteq'$, denoted by $\mathcal{D}(\trianglelefteq, \trianglelefteq')$, is the cardinality of their symmetric difference. In symbols:

$$\mathcal{D}(\trianglelefteq, \trianglelefteq') = \Big| \big(\trianglelefteq - \trianglelefteq' \big) \cup \big(\trianglelefteq' - \trianglelefteq \big) \Big|.$$

Example 3. Assume that $\mathcal{P} = \{a, b, c, d\}$, and let $\trianglelefteq, \trianglelefteq'$ be the following total preorders over $\mathcal{P}$: $a \triangleleft b \sim c \triangleleft d$ and $a \sim' c \triangleleft' b \sim' d$. Then, $\mathcal{D}(\trianglelefteq, \trianglelefteq') = 3$.

Given an arbitrary threshold $\mathcal{CR}$ opted by the agent, several *substitution policies* (rules) can be established.[12] Consider, indicatively, the two presented below:

Substitution Policy I:	The preorder $\trianglelefteq$ should be substituted by the preorder $\trianglelefteq^{\varphi}$ iff $\mathcal{D}(\trianglelefteq, \trianglelefteq^{\varphi}) > \mathcal{CR}$.
Substitution Policy II:	The preorder $\trianglelefteq$ should *not* be substituted by the preorder $\trianglelefteq^{\varphi}$ iff $\mathcal{D}(\trianglelefteq, \trianglelefteq^{\varphi}) > \mathcal{CR}$.

Both the above rules are plausible depending on the underlying application or domain to be encoded. Note that the second substitution policy is aligned with the intuition in some approaches to *non-prioritized* revision, where major changes are rejected; in particular, when the new information φ requires "a great degree" of change to the initial belief set, φ is ignored [15, 16].

The process of dynamic PD revision is illustrated in Figure 3, for a critical epistemic input φ. Observe that the whole procedure of replacing $\trianglelefteq$ with $\trianglelefteq^{\varphi}$ is implemented with *no extra representational cost*; the only information required is the new information φ. Therefore, the benefits of PD operators concerning compact specification are totally preserved. Another plausible scenario would be the change of $\trianglelefteq$ be firing by the revised belief set $K * \varphi$, and not by φ itself. Of course, the aforementioned analysis for dynamic PD revision applies for such a case as well; the only difference is that the function ρ takes $K * \varphi$, instead of φ, as its first argument.

It is implied from Figure 3 that dynamic PD revision could lead to a change in the original definition of revision functions (Section 3). In particular, revision functions may become

[12] The use of an arbitrary (quantitative) threshold, opted by the agent, would give flexibility and adaptability to a real-world PD-revision system, the implementation of which is the ultimate purpose of this line of research.

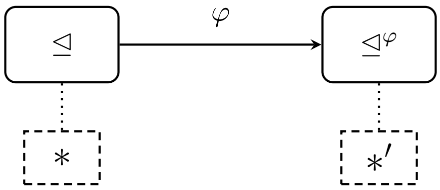

FIGURE 3. The process of dynamic PD revision. Formula φ is critical and causes a substitution of the preorder $\trianglelefteq$ (which induces the PD operator $*$) by the preorder $\trianglelefteq^{\varphi}$ (which induces the PD operator $*'$).

dynamic in nature, in the sense that a revision function may change as new evidence arrives. To see this, consider an agent whose initial belief set is K_0. Suppose that, after a sequence of revisions by a sequence of sentences $\varphi_1, \varphi_2, \ldots, \varphi_n$, the agents ends up with a belief set K_n, such that $K_n = K_0$. With dynamic revision functions, it is possible that one revision function is associated with K_0, and a different one with K_n. On that premise, we are allowed to define a (binary) revision function $*$ as a *unary* function that maps an epistemic input φ to a new belief set $*_K(\varphi)$, given an initial belief set K as background.[13]

After the definition of dynamic PD revision, we proceed to the study of PD revision in Horn logic.

7. PD Horn Revision

Given that Horn logic (namely, the Horn fragment of propositional logic) has been applied numerous times in both Artificial Intelligence and databases, as well as that the computational-complexity properties of PD operators restricted to Horn logic are quite compelling, the study of PD Horn revision becomes imperative; for an extensive study on classical AGM-style Horn revision, the interested reader is referred to the work of Delgrande and Peppas [11]. For the study of PD Horn revision, some basic notation and terminology are necessary, introduced subsequently.

A clause (i.e., a disjunction of literals) is called a *Horn clause* iff it contains *at most* one positive literal; e.g., $a \vee \neg b \vee \neg c$. Horn clauses are usually written as

$$a_1 \wedge a_2 \wedge \cdots \wedge a_n \rightarrow a,$$

which is logically equivalent to

$$\neg a_1 \vee \neg a_2 \vee \cdots \vee \neg a_n \vee a,$$

where $n \geqslant 0$ and a is the *only* positive literal. Whenever $n = 0$, then $\rightarrow a$ is written as a, and it is called *fact*.

A *Horn formula* is the conjunction of Horn clauses. The *Horn language* $\mathcal{L}_H$ is the maximal subset of $\mathcal{L}$ containing only Horn formulas. The letter H is reserved to represent a Horn theory

[13] Interestingly, the aforementioned reformulations of revision functions —which do not require any alteration of the AGM postulates for revision, since all of them refer only to a single theory of $\mathcal{L}$— has been proposed by Nayak *et al.* [23], as a way to reconcile the AGM paradigm with the original Darwiche and Pearl's approach for iterated belief revision [10].

(belief set). The set of all Horn theories is denoted by $\mathbb{H}$. The Horn logic generated from $\mathcal{L}_H$ is specified by the consequence operator Cn_H, such that, for any set of Horn formulas Γ,

$$Cn_H(\Gamma) = Cn(\Gamma) \cap \mathcal{L}_H.$$

Consider, now, the following definition.

Definition 9 (Positive-Literals Intersection of Worlds)**.** Let r, r' be two possible worlds of $\mathbb{M}$. The positive-literals intersection of r, r', denoted by $r \cap^+ r'$, is the possible world defined as follows:

$$r \cap^+ r' = (r \cap r')^+ \cup \left(\overline{\mathcal{P} - (r \cap r')^+}\right).$$

For instance, the positive-literals intersection of the worlds $\{a, b, c\}$ and $\{a, \neg b, \neg c\}$ is the world $\{a, \neg b, \neg c\}$, whereas, the positive-literals intersection of the worlds $\{\neg a, b, c\}$ and $\{a, \neg b, \neg c\}$ is the world $\{\neg a, \neg b, \neg c\}$.

An arbitrary formula (or theory) φ is Horn (i.e., $\varphi \in \mathcal{L}_H$) iff the possible worlds satisfied by φ are closed under intersection of positive literals, i.e., whenever $r, r' \in [\varphi]$ entails $r \cap^+ r' \in [\varphi]$. The previous statement can be expressed by means of the notion of an *elementary* set of worlds as well. In particular, we shall say that a non-empty set W of possible worlds is elementary iff

$$\left\{r \cap^+ r' : \text{for all } r, r' \in W\right\} = W.$$

Then, we can state that an arbitrary formula (or theory) φ is Horn iff the set $[\varphi]$ of worlds is elementary.

7.1. **Horn Compliant Preorders.** An AGM Horn revision function $*$ is an AGM revision function that maps a Horn theory and a Horn formula to a (new) Horn theory; i.e., $* : \mathbb{H} \times \mathcal{L}_H \mapsto \mathbb{H}$. With the aid of elementary sets of worlds, we can define exactly those faithful preorders that induce (via condition (F$*$)) AGM Horn revision functions, called *Horn compliant* preorders.

Definition 10 (Horn Compliant Preorder, [33])**.** A preorder $\preceq_H$ (faithful to a Horn theory H) is Horn compliant iff, for every Horn formula $\varphi \in \mathcal{L}_H$, the set of worlds $min([\varphi], \preceq_H)$ is elementary.

Zhuang and Pagnucco provided a (faithful-preorders) characterization of *Horn compliance*, by means of the following constraint on possible worlds [33]:

(H) If $r \approx r'$, then $r \cap^+ r' \preceq r$.

Condition (H) says that whenever two worlds r and r' are equidistant from the beginning of a preorder $\preceq$, then the world $r \cap^+ r'$, resulting from their positive-literals intersection, cannot appear later in $\preceq$.

Zhuang and Pagnucco showed that any preorder that satisfies condition (H) is Horn compliant, and, conversely, any Horn compliant preorder satisfies (H) [33].[14]

[14]For an axiomatic (postulational) characterization of AGM Horn revision functions, the reader is referred to [11].

7.2. **PD Operators are not Horn Compliant.** Although the computational complexity of PD revision was extensively studied in [27], it was not shown whether PD operators are, indeed, compatible with respect to Horn revision; that is to say, whether the PD-revision of a Horn knowledge base by a Horn formula *always* yields a (new) Horn knowledge base. Unfortunately, the subsequent result shows that PD operators, as originally defined in [26, 27], are *not* Horn compliant.

Theorem 2. *There exists a Horn theory H for which a PD preorder, associated with H, is not Horn compliant.*

Proof. Assume that $\mathcal{L}$ is built from $\mathcal{P} = \{a, b, c\}$, and let H be the Horn theory $H = Cn_H(a, b, c)$. Moreover, let $\trianglelefteq$ be a total preorder over the set of atoms $\mathcal{P}$, such that $c \lhd a \sim b$. Then, the (only) PD preorder $\sqsubseteq_H^{\trianglelefteq}$ assigned to H is the following:

$$abc \quad \sqsubset_H^{\lhd} \quad ab\overline{c} \quad \sqsubset_H^{\lhd} \quad \begin{matrix} \overline{a}bc \\ a\overline{b}c \end{matrix} \quad \sqsubset_H^{\lhd} \quad \begin{matrix} \overline{a}b\overline{c} \\ a\overline{b}\overline{c} \end{matrix} \quad \sqsubset_H^{\lhd} \quad \overline{a}\overline{b}c \quad \sqsubset_H^{\lhd} \quad \overline{a}\overline{b}\overline{c}$$

Notice, however, that for the worlds $r = \overline{a}bc$ and $r' = a\overline{b}c$, which they are equally plausible with respect to $\sqsubseteq_H^{\trianglelefteq}$, it is true that $r \cap^+ r' = \overline{a}\overline{b}c$. Therefore, the worlds r, r', $r \cap^+ r'$ violate condition (H), and, hence, $\sqsubseteq_H^{\trianglelefteq}$ is not Horn compliant. □

Of course, the above impossibility result does not entail that the PD-revision of a Horn knowledge base by a Horn formula —which, as stated, is characterized by low computational cost— is not a legitimate operation. All that Horn non-compliance entails is that the result of the revision *may be* a non-Horn knowledge base. For instance, given an epistemic input φ such that $[\varphi] = \{\overline{a}bc, a\overline{b}c\}$, the preorder in the proof of Theorem 2 (partially) induces a PD operator $*$, such that $[H * \varphi] = \{\overline{a}bc, a\overline{b}c\}$; thus, $H * \varphi$ is a non-Horn theory. However, for an epistemic input ψ such that $[\psi] = \{ab\overline{c}, \overline{a}bc\}$, $[H * \psi] = \{ab\overline{c}\}$; that is, $H * \psi$ is indeed a Horn theory.[15]

Against this non-satisfactory background, an alternative way for defining Horn compliant revision operators, parametrized by a preorder over atoms, is presented in Subsection 7.4, based on the notion of *basic Horn revision* introduced in the next subsection. Furthermore, on top of very recent results on extended languages [5], an indirect connection between PD and Horn revision is established in Subsection 7.5.

7.3. **Basic Horn Revision.** In [11], it was shown that Dalal's and Satoh's operators cannot, directly, be applied in a Horn setting. To remedy this weakness, the authors proposed their own Horn compliant revision operator, inspired by the *atom-based* plausibility of the original work of Dalal and Satoh.

In particular, let H be a Horn belief set. Consider, then, the following *basic Horn ranking* $\preceq_H$ over all possible worlds of $\mathbb{M}$:

(BH) $\quad r \preceq_H r' \quad$ iff $\quad$ either $r \in [H] \quad$ or $r, r' \notin [H]$ and $|r^+| \leqslant |r'^+|$.

This ordering reflects the intuition that an atom is false, unless it is "required" to be true. In this case, the authors give preference to worlds with *fewer true* atomic propositions.

Definition 11 (Basic Horn Revision). The basic Horn revision function $\diamond$ is defined to be the revision function induced from the family of basic Horn rankings $\{\preceq_H\}_{H \in \mathbb{H}}$, via condition (F$*$).

[15]We suspect that an incompatibility result stronger than Theorem 2 cannot be achieved.

As proved in [11], the preorder $\preceq_H$ is a total preorder faithful to H (for any $H \in \mathbb{H}$), therefore, in view of Theorem 1, $\diamond$ satisfies $(K * 1)$–$(K * 8)$ at H. It was shown, moreover, that $\diamond$ has nice computational properties, as it can be computed in *linear time*. Nevertheless, basic Horn revision has a (sometimes undesirable) feature; whenever the epistemic input φ is inconsistent with the initial belief set H, the revised belief $H \diamond \varphi$ is *always complete*.

In cases where completeness can be tolerated, it appears that basic Horn revision is a satisfactory proposal. Yet, the basic Horn ranking $\preceq_H$, although Horn compliant, is in some sense counter-intuitive. We shall illustrate this with the aid of a concrete example. Suppose that $\mathcal{L}$ is built from $\mathcal{P} = \{a, b\}$, and consider the Horn belief set $H = Cn_H(a, b)$. Then, the only basic Horn ranking $\preceq_H$ that can be assigned to H is the following:

$$ab \quad \prec_H \quad \overline{a}\overline{b} \quad \prec_H \quad \begin{matrix} \overline{a}b \\ a\overline{b} \end{matrix}$$

Notice that, although we initially believe that both a and b are true, the world $\overline{a}\overline{b}$ appears more plausible than both worlds $\overline{a}b$ and $a\overline{b}$, even though the former differs from ab in more propositional variables than the latter two. The main reason for this counter-intuitive behaviour of basic Horn ranking is that, in order to construct the preorder $\preceq_H$, the initial belief set H of the agent is not taken into consideration; the worlds outside $[H]$ are ranked solely depending on their true atomic propositions —an intrinsic feature of any possible world— and not on some notion of *difference* from H.

7.4. **Parametrized Basic Horn Revision.** Inspired by the approach of Delgrande and Peppas, we develop our own *parametrized* Horn revision operator, based on the cardinality of positive literals of worlds. The operator we propose is, essentially, a $\trianglelefteq$-parametrized version of the basic Horn revision function.

To this end, let H be a Horn belief set, and let $\trianglelefteq$ be a total preorder over the set of propositional variables $\mathcal{P}$. We define, then, the *parametrized basic Horn ranking* $\preceq_H^{\trianglelefteq}$ over the possible worlds of $\mathbb{M}$, as follows:

(PBH) $\quad r \preceq_H^{\trianglelefteq} r'$ iff either $r \in [H]$ or $r, r' \notin [H]$ and $r^+ \trianglelefteq r'^+$.

The next theorem shows that $\preceq_H^{\trianglelefteq}$ is a faithful-to-H preorder, and, moreover, Horn-compliant.

Theorem 3. *Let H be a Horn belief set, let $\trianglelefteq$ be a total preorder over the set of propositional variables $\mathcal{P}$, and let $\preceq_H^{\trianglelefteq}$ be a parametrized basic Horn ranking associated with H. Then, $\preceq_H^{\trianglelefteq}$ is a total preorder, faithful to H, and Horn compliant.*

Proof. By definition, $\preceq_H^{\trianglelefteq}$ is reflexive. For transitivity, let r, r', r'' be any three worlds of $\mathbb{M}$, such that $r \preceq_H^{\trianglelefteq} r'$ and $r \preceq_H^{\trianglelefteq} r'$. If $r \in [H]$, then clearly $r \preceq_H^{\trianglelefteq} r''$. Assume, therefore, that $r \notin [H]$. Then, $r \preceq_H^{\trianglelefteq} r'$ entails $r' \notin [H]$ and $r^+ \trianglelefteq r'^+$. Similarly, from $r' \notin [H]$ and $r' \preceq_H^{\trianglelefteq} r''$, we derive that $r'' \notin [H]$ and $r'^+ \trianglelefteq r''^+$. Hence, $r, r'' \notin [H]$ and $r^+ \trianglelefteq r''^+$. Consequently, $r \preceq_H^{\trianglelefteq} r''$, as desired.

For totality, let r and r' be any two worlds of $\mathbb{M}$. If either of them is in $[H]$, then clearly the two worlds are comparable with respect to $\preceq_H^{\trianglelefteq}$. Assume, therefore, that neither of them belongs to $[H]$. If $r^+ \trianglelefteq r'^+$, then $r \preceq_H^{\trianglelefteq} r'$; otherwise, $r' \preceq_H^{\trianglelefteq} r$.[16] In either case, r, r' are comparable with respect to $\preceq_H^{\trianglelefteq}$, and, hence, $\preceq_H^{\trianglelefteq}$ is total.

[16] Recall that the preorder $\trianglelefteq$ is total.

Faithfulness with respect to H follows immediately from the definition of $\preceq_H^{\trianglelefteq}$. Hence, to complete the proof, we need to show that $\preceq_H^{\trianglelefteq}$ is Horn compliant, by proving that condition (H) is satisfied. Consider, therefore, any two worlds r, r' of $\mathbb{M}$, such that $r \approx_H^{\trianglelefteq} r'$. We will show that $r \cap^+ r' \preceq_H^{\trianglelefteq} r$. If $r \cap^+ r' \in [H]$ or $r = r'$, this is clearly true. Assume, therefore, that $r \cap^+ r' \notin [H]$ and, moreover, $r \neq r'$. Then, since H is a Horn theory, not both r and r' can be members of $[H]$. Since one of r, r' is not in $[H]$, from $r \approx_H^{\trianglelefteq} r'$, we derive that neither of the two worlds belongs to $[H]$. From $r \approx_H^{\trianglelefteq} r'$, we also derive that $|r^+| = |r'^+|$. Hence, given that $r \neq r'$, it is not hard to verify that $|r \cap^+ r'| < |r^+|$; thus, by the definition of $\trianglelefteq$ (Definition 5), we have that $r \cap^+ r' \trianglelefteq r^+$. Consequently, $r \cap^+ r' \preceq_H^{\trianglelefteq} r$, as desired. □

Observe that the parametrized basic Horn ranking $\preceq_H^{\trianglelefteq}$, such as the basic Horn ranking $\preceq_H$, does not take into account some notion of *difference* of worlds from H. Hence, the following remark is clearly true.

Remark 1. *The preorder $\preceq_H^{\trianglelefteq}$ is a parametrized ranking, but not a parametrized-difference ranking.*

Parametrized basic Horn revision is defined subsequently.

Definition 12 (Parametrized Basic Horn Revision)**.** Let $\trianglelefteq$ be a total preorder over $\mathcal{P}$. We define the parametrized basic Horn revision function $\star$ to be the revision function induced from the family of parametrized basic Horn rankings $\{\preceq_H\}_{H \in \mathbb{H}}$, via condition (F$*$).

As $\preceq_H^{\trianglelefteq}$ is faithful to H (for any Horn theory $H \in \mathbb{H}$), in view of Theorem 1, $\star$ satisfies $(K*1)$–$(K*8)$ at H.

Let us, now, introduce the following definition, which will help us to establish an important observation concerning the relation between the revision functions $\star$ and $\diamond$.

Definition 13 (Horn-Revision-Equivalent Operators)**.** We shall say that two revision functions $*$ and $*'$ are Horn-revision-equivalent iff, for any Horn theory H of $\mathcal{L}_H$ and all Horn formulas $\varphi \in \mathcal{L}_H$, $H * \varphi = H *' \varphi$.

It has been proven in Proposition 5 of [11, p. 18] that, for every consistent Horn formula φ, there is *only one* φ-world, such that it has the least number of positive literals. This observation implies the following result.

Proposition 1. *The parametrized basic Horn revision function $\star$ is Horn-revision-equivalent to the basic Horn revision function $\diamond$.*

Proof. Follows immediately from condition (i) of Definition 5, conditions (BH) and (PBH), and Proposition 5 of [11, p. 18]. □

Consequently, the parametrization preorder $\trianglelefteq$ becomes redundant, and $\star$ degenerates to $\diamond$, with respect to Horn revision.

In the remainder of this section, a rigid connection between PD and Horn revision is proved, on top of very recent results on extended languages established by the author (along with Peppas and Williams) [5].

7.5. **Connecting PD and Horn Revision via Language Extension.** As showed in Subsection 7.2, PD operators are not Horn compliant. At first glance, this seems to be a conclusive impossibility result. Nevertheless, the very recent results on extended languages of [5] offer an appealing different direction for relating PD and Horn revision. In particular, it was shown in

[5] that *any* total preorder over the worlds of $\mathbb{M}$ can be "extracted" from a Dalal's preorder, defined at a sufficiently *extended* language.[17] Although we shall not present the formal process of this "extraction" herein (it is in detail described in [5]), its informal intuition is illustrated in the following concrete example.

Example 4 ([5])**.** Suppose that $\mathcal{L}$ is built from $\mathcal{P} = \{a, b\}$. Moreover, let K be a (complete) theory of $\mathcal{L}$, such that $K = Cn(a, b)$. Then, the (only) Dalal's preorder $\sqsubseteq_K$ that is assigned at K is the following:

$$ab \quad \sqsubset_K \quad \begin{matrix} a\overline{b} \\ \overline{a}b \end{matrix} \quad \sqsubset_K \quad \overline{a}\overline{b}$$

Suppose, now, that we want to assign at theory K the following non-Dalal's preorder $\preceq_K$:

$$ab \quad \prec_K \quad \overline{a}b \quad \prec_K \quad a\overline{b} \quad \prec_K \quad \overline{a}\overline{b}$$

Consider the set of worlds $S = \{abc, \overline{a}bc, a\overline{b}\overline{c}, \overline{a}\overline{b}\overline{c}\}$ (where c is an atom not included in $\mathcal{L}$), and the following restricted Dalal's preorder $\sqsubseteq$ over S:

$$abc \quad \sqsubset \quad \overline{a}bc \quad \sqsubset \quad a\overline{b}\overline{c} \quad \sqsubset \quad \overline{a}\overline{b}\overline{c}$$

Clearly, if we ignore the propositional variables of the worlds of S that are outside $\mathcal{L}$, we get the desired preorder $\preceq_K$.

Note that $\sqsubseteq$ is a restriction (part) of the next Dalal's preorder $\sqsubseteq_{K'}$, associated with the (complete) theory $K' = Cn(a, b, c)$:

$$abc \quad \sqsubset_{K'} \quad \begin{matrix} \overline{a}bc \\ a\overline{b}c \\ ab\overline{c} \end{matrix} \quad \sqsubset_{K'} \quad \begin{matrix} \overline{a}\overline{b}c \\ a\overline{b}\overline{c} \\ \overline{a}b\overline{c} \end{matrix} \quad \sqsubset_{K'} \quad \overline{a}\overline{b}\overline{c}$$

We shall say that the non-Dalal's preorder $\preceq_K$ of the above example constitutes a *filtering* of the Dalal's preorder $\sqsubseteq_{K'}$.

Given that a Dalal's preorder is a particular PD preorder (i.e., in case $\trianglelefteq = \mathcal{P} \times \mathcal{P}$), the following important theorem is obviously true.

Theorem 4. *Any Horn compliant preorder, defined at $\mathcal{L}$, constitutes a filtering of some PD preorder, defined at a sufficient extension of $\mathcal{L}$.*

Proof. Obvious from Theorem 3 of [5], and the fact that a Dalal's preorder is a particular PD preorder, in case $\trianglelefteq = \mathcal{P} \times \mathcal{P}$. □

Theorem 4, essentially, implies that PD revision and Horn revision are, in fact, *strongly related*; *any* Horn compliant preorder, defined at $\mathcal{L}$, is a fragment of some richer PD preorder. In a way, a Horn compliant preorder at $\mathcal{L}$ constitutes an epiphenomenon of an underlying richer PD preorder, as the (limited) expressivity of language $\mathcal{L}$ does not permit the formulation of the latter.

Having analysed the relation between PD and Horn revision, in what follows, we turn to the study of PD revision in the realm of Description Logics (DL).

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[17]A propositional language is called an *extension* of $\mathcal{L}$ iff it is finitary, and, moreover, it is built from a set of atoms that is a proper superset of $\mathcal{P}$.

8. PD Revision in Description Logics

This section examines an approach of the DP-revision problem in the realm of an important knowledge-representation formalism, namely, *Description Logics* (DL). DL are families of knowledge representation languages —in fact, *decidable* fragments of first-order logic— that are widely used in ontological modelling and the Semantic Web [17].[18] The name "Description Logics" is motivated by the fact that the important notions of the domain to be encoded are described by *concept descriptions*, namely, expressions that are built from atomic *concepts* (unary predicates) and atomic *roles* (binary predicates), using the concept and role *constructors* provided by the particular description logic [8].

A DL knowledge base is made up of two components, a *terminological* part, called *TBox*, and an *assertional* part, called *ABox*, each part consisting of a set of *axioms*. A TBox consists of *concept axioms* and *role axioms*, and essentially describes the relevant notions of an application domain, by stating properties of concepts and roles, as well as relationships between them. On the other hand, an ABox is used to describe a concrete situation by stating *properties of individuals* [8]. An example of a TBox is

$$WOMAN \sqsubseteq HUMAN$$

which states that all women are humans. An example of an ABox is

$$WOMAN(MARY)$$

which states that the individual Mary is a woman. Note that, in what follows, we shall confine ourselves to the revision of only terminological knowledge; i.e., TBoxes.

DL are extensively used in Artificial Intelligence, as well as in domains such as Software Engineering, Medicine, Database Management, Planning and Data Mining, for modelling an application domain in a concrete and structured way, and for providing reasoning tools that can deduce implicit from explicit knowledge. Several *inference problems* are of great interest in DL, such as database-query-like questions like *instance checking* (whether a given individual is instance of a specified concept description), and global-database-questions like *subsumption* (whether a concept is a subset of another concept).

A variety of DL has been considered in the literature, according to which constructors —such as conjunction of concepts, disjunction of concepts, negation of concepts, existential restriction, universal restriction— are allowed to be used to form the syntax of the language. Different combinations of constructors result in a different trade-off between the *expressivity* of DL and the *complexity* of their inference problems. This exact balance constitutes an important research topic, from both a theoretical and an applied perspective; the golden mean, however, depends on the intended application. Furthermore, several language extensions have been investigated, which have given rise to DL addressing non-monotonic, epistemic, spatio-temporal and fuzzy reasoning, as well as to DL that are able to represent uncertain and vague knowledge [7].

For an excellent survey on the broad topic of DL, the interested reader is referred to [8].

8.1. **Notation and Terminology of Description Logics.** A TBox consists of *concept axioms* of the form $A \sqsubseteq B$, where A, B are (possibly complex) concept expressions, and *role axioms* of the form $R \sqsubseteq S$, where R, S are (possibly complex) role expressions. The set of all concept

[18]Well-known examples of DL-based ontology languages include Ontology Inference Layer (OIL), DARPA Agent Markup Language +OIL, and Web Ontology Language (OWL).

names is denoted by N_C, and the set of all role names is denoted by N_R. The set of all concept and role names is denoted by $\mathcal{N}$; i.e.,

$$\mathcal{N} = N_C \cup N_R.$$

Definition 14 (Interpretation)**.** An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a non-empty set $\Delta^{\mathcal{I}}$, called the domain of $\mathcal{I}$, and a function $\cdot^{\mathcal{I}}$ that maps every concept description to a subset of $\Delta^{\mathcal{I}}$, and every role name to a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.

The set of all interpretations in the considered DL language is denoted by Ω. For ease of presentation, we shall assume that all interpretations are defined in a *common* domain Δ.

An interpretation $\mathcal{I}$ is a *model* of an axiom $A \sqsubseteq B$ iff $A^{\mathcal{I}} \subseteq B^{\mathcal{I}}$. $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$, written $\mathcal{I} \models \mathcal{T}$, iff it is a model of every axiom in $\mathcal{T}$. For a TBox $\mathcal{T}$, $[\mathcal{T}]$ denotes the set of all models of $\mathcal{T}$. Notation $A \doteq B$ abbreviates $A \sqsubseteq B$ and $B \sqsubseteq A$.

8.2. **Postulates for Revision Operators in Description Logics.** Qi *et al.* reformulated Katsuno and Mendelzon's postulates (KM1)–(KM6) of Section 3 in the DL setting, adopting a *model-based* approach [28]. To this end, they defined a *DL revision operator* to be a (binary) function mapping a pair of TBoxes to a *disjunctive* Tbox. A disjunctive TBox, originally defined in [22], is a set of TBoxes. An interpretation is a model of a disjunctive TBox T iff it is a model of one of the TBoxes in T. Against this background, and given the TBoxes $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{T}$, $\mathcal{T}'$, the resulting set of postulates (G1)–(G6) is listed below; note that the symbol $\circ$ shall be used for DL revision operator operating on TBoxes.[19]

- **(G1)** $[\mathcal{T}_1 \circ \mathcal{T}_2] \subseteq [\alpha]$, for all $\alpha \in \mathcal{T}_2$.
- **(G2)** If $[\mathcal{T}_1] \cap [\mathcal{T}_2] \neq \varnothing$, then $[\mathcal{T}_1 \circ \mathcal{T}_2] = [\mathcal{T}_1] \cap [\mathcal{T}_2]$.
- **(G3)** If $\mathcal{T}_2$ is consistent, then $[\mathcal{T}_1 \circ \mathcal{T}_2] \neq \varnothing$.
- **(G4)** If $[\mathcal{T}] = [\mathcal{T}']$ and $[\mathcal{T}_1] = [\mathcal{T}_2]$, then $[\mathcal{T} \circ \mathcal{T}_1] = [\mathcal{T}' \circ \mathcal{T}_2]$.
- **(G5)** $[\mathcal{T}_1 \circ \mathcal{T}_2] \cap [\mathcal{T}_3] \subseteq \big[\mathcal{T}_1 \circ (\mathcal{T}_2 \cup \mathcal{T}_3)\big]$.
- **(G6)** If $[\mathcal{T}_1 \circ \mathcal{T}_2] \cap [\mathcal{T}_3] \neq \varnothing$, then $\big[\mathcal{T}_1 \circ (\mathcal{T}_2 \cup \mathcal{T}_3)\big] \subseteq [\mathcal{T}_1 \circ \mathcal{T}_2] \cap [\mathcal{T}_3]$.

The interpretation of the above postulates is as follows [30]: (G1) guarantees that every axiom in the new TBox can be inferred from the result of revision. (G2) says that we do not change the original TBox if there is no conflict. (G3) is a condition preventing a revision from introducing unwarranted inconsistency. (G4) says that the DL revision operator should be independent of the syntactical forms of TBoxes. Lastly, (G5) and (G6), together, ensure minimal change.

Qi *et al.* provide a representation theorem for postulates (G1)–(G6), with respect to a refinement of the notion of faithful preorder.

Definition 15 (Faithful Preorder over Interpretations, [29])**.** Let $\mathcal{T}$, $\mathcal{T}'$ be two TBoxes. A preorder $\preceq_{\mathcal{T}}$ over Ω is faithful to $\mathcal{T}$ iff it is total, and such that, for any two interpretations $\mathcal{I}$, $\mathcal{I}' \in \Omega$:

- **(i)** If $\mathcal{I} \models \mathcal{T}$, then $\mathcal{I} \preceq_{\mathcal{T}} \mathcal{I}'$.
- **(ii)** If $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I}' \not\models \mathcal{T}$, then $\mathcal{I} \prec_{\mathcal{T}} \mathcal{I}'$.

[19]The reformulated postulates of Qi *et al.* presented herein are confined to TBoxes.

(iii) If $\mathcal{T} \doteq \mathcal{T}'$, then $\preceq_{\mathcal{T}} = \preceq_{\mathcal{T}'}$.

The following representation theorem resembles Theorem 1 of Katsuno and Mendelzon (Section 3), and establishes the correspondence between the set of postulates (G1)–(G6) and faithful preorders over interpretations.

Theorem 5 ([29])**.** *A DL revision operator* $\circ$ *satisfies postulates (G1)–(G6) iff, for any TBoxes* $\mathcal{T}, \mathcal{T}'$*, there exists a faithful preorder* $\preceq_{\mathcal{T}}$*, such that:*

$$\textbf{(F}\circ\textbf{)} \quad [\mathcal{T} \circ \mathcal{T}'] = min([\mathcal{T}'], \preceq_{\mathcal{T}}).$$

Condition (F$\circ$) says that the models of the revised TBox $\mathcal{T} \circ \mathcal{T}'$ are exactly the $\preceq_{\mathcal{T}}$-minimal models of the TBox $\mathcal{T}'$.

8.3. **Hamming-Based DL Revision Operators.** The purpose of this section is the introduction of *Hamming-based* DL revision operators, inspired by Dalal's proposal in the propositional setting. Accordingly, we define subsequently a notion of *difference* between interpretations.

Definition 16 (Difference between Interpretations)**.** Let $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$, $\mathcal{I}' = (\Delta, \cdot^{\mathcal{I}'})$ be two interpretations of Ω. The difference between $\mathcal{I}$ and $\mathcal{I}'$, denoted by $\mathit{Diff}\,(\mathcal{I}, \mathcal{I}')$, is as follows:

$$\mathit{Diff}\,(\mathcal{I}, \mathcal{I}') = \Big\{ N \in \mathcal{N} : N^{\mathcal{I}} \neq N^{\mathcal{I}'} \Big\}.$$

The above definition mirrors the notion of Hamming distance, and says that the difference between any two interpretations $\mathcal{I}$ and $\mathcal{I}'$ is the set of all concept and role names that are interpreted differently by the interpretations. Observe that, if $\mathcal{I}$, $\mathcal{I}'$ do not share a common domain, then they are *Diff-incomparable*.

Example 5. Let $N_C = \{A, B\}$ and $N_R = \{r, s\}$. Moreover, let $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$, $\mathcal{I}' = (\Delta, \cdot^{\mathcal{I}'})$ be two interpretations of Ω, such that:

- $\Delta = \{a, b, c, d\}$,
- $A^{\mathcal{I}} = \{a, b\}$, $B^{\mathcal{I}} = \{a, b\}$, $r^{\mathcal{I}} = \big\{\langle a, c\rangle\big\}$, $s^{\mathcal{I}} = \big\{\langle c, d\rangle\big\}$, and
- $A^{\mathcal{I}'} = \{a, b, c\}$, $B^{\mathcal{I}'} = \{a, b\}$, $r^{\mathcal{I}'} = \big\{\langle a, c\rangle\big\}$, $s^{\mathcal{I}'} = \big\{\langle c, d\rangle, \langle a, d\rangle\big\}$.

Since $A^{\mathcal{I}} \neq A^{\mathcal{I}'}$ and $s^{\mathcal{I}} \neq s^{\mathcal{I}'}$, Definition 16 implies that $\mathit{Diff}\,(\mathcal{I}, \mathcal{I}') = \{A, s\}$.

In the DL setting, other more sophisticated notions of difference between interpretations could be deployed, providing higher levels of granularity. For instance, the difference between any two interpretations $\mathcal{I}$, $\mathcal{I}'$, can be defined with respect to a concept or role name interpreted by $\mathcal{I}$, $\mathcal{I}'$, based on *symmetric difference*.

Definition 17 (N-Difference)**.** Let $\mathcal{I}$, $\mathcal{I}'$ be two interpretations of Ω. For any concept/role name $N \in \mathcal{N}$, the N-difference between $\mathcal{I}$ and $\mathcal{I}'$, denoted by $\mathit{Diff}_N(\mathcal{I}, \mathcal{I}')$, is as follows:

$$\mathit{Diff}_N(\mathcal{I}, \mathcal{I}') = N^{\mathcal{I}} \ominus N^{\mathcal{I}'},$$

where $N^{\mathcal{I}} \ominus N^{\mathcal{I}'}$ denotes the symmetric difference of the sets $N^{\mathcal{I}}$, $N^{\mathcal{I}'}$.

Utilizing the above definition, we can deploy the following *quantitative* measure of difference, inspired by [32]:

Definition 18 (Quantitative Difference between Interpretations)**.** Let $\mathcal{I}$, $\mathcal{I}'$ be two interpretations of Ω. The quantitative difference between $\mathcal{I}$ and $\mathcal{I}'$, denoted by $Diff'(\mathcal{I},\mathcal{I}')$, is as follows:

$$Diff'(\mathcal{I},\mathcal{I}') = \sum_{N \in \mathcal{N}} \big|Diff_N(\mathcal{I},\mathcal{I}')\big|.$$

Based on the difference of Definition 16, we define the *distance* between a TBox $\mathcal{T}$ and an interpretation $\mathcal{I}$ as follows:

Definition 19 (Distance between a TBox and an Interpretation)**.** Let $\mathcal{T}$ be a TBox and let $\mathcal{I}$ be an interpretation of Ω. The distance between $\mathcal{T}$ and $\mathcal{I}$, denoted by $Dist\,(\mathcal{T},\mathcal{I})$, is as follows:

$$Dist\,(\mathcal{T},\mathcal{I}) = min\Big(\Big\{\big|Diff\,(\mathcal{I}',\mathcal{I})\big| : \mathcal{I}' \in [\mathcal{T}]\Big\}\Big).$$

The use of the difference of Definition 18 in the preceding notion of distance is also feasible, with the replacement of the term "$|Diff\,(\mathcal{I}',\mathcal{I})|$" with the term "$Diff'(\mathcal{I}',\mathcal{I})$".

Now, let $\mathcal{T}$ be a TBox and $\mathcal{I}$, $\mathcal{I}'$ be any two interpretations, and let $\preceq_{\mathcal{T}}$ be a ranking associated with $\mathcal{T}$, defined as follows:

(D1) $\mathcal{I} \preceq_{\mathcal{T}} \mathcal{I}'$ iff $Dist\,(\mathcal{T},\mathcal{I}) \leqslant Dist\,(\mathcal{T},\mathcal{I}')$.

It is not hard to verify that the ranking $\preceq_{\mathcal{T}}$, defined by means of condition (D1), is a total preorder, faithful to $\mathcal{T}$. Therefore, from Theorem 5, the following result is obtained.

Proposition 2. *The DL revision operators induced by means of condition (D1), via (F$\circ$), satisfy postulates (G1)–(G6).*

Apart from the "off-the-shelf" DL revision operator defined via condition (D1), several constraints on a preorder $\preceq_{\mathcal{T}}$ faithful to a TBox $\mathcal{T}$ can be applied, based on *set inclusion* as well. Consider, for instance, the following condition:

(D2) If $Diff_N(\mathcal{W},\mathcal{I}) \subseteq Diff_N(\mathcal{W},\mathcal{I}')$, for every $\mathcal{W} \in [\mathcal{T}]$ and every $N \in \mathcal{N}$, then $\mathcal{I} \preceq_{\mathcal{T}} \mathcal{I}'$.

Condition (D2) states that, if, for every model $\mathcal{W}$ of $\mathcal{T}$ and every concept/role name N, the N-difference between $\mathcal{W}$ and $\mathcal{I}$ is a subset of the N-difference between $\mathcal{W}$ and $\mathcal{I}'$, then $\mathcal{I}$ ought to be at least as plausible as $\mathcal{I}'$, with respect to $\mathcal{T}$. The next example contrasts the constraints (D1) and (D2).

Example 6. Consider the scenario of Example 5, and let $\mathcal{T}$ be a TBox such that it has a single model $\mathcal{W}$, for which $A^{\mathcal{W}} = \{a\}$, $B^{\mathcal{W}} = \{a,b\}$, $r^{\mathcal{W}} = \big\{\langle b,c\rangle\big\}$, $s^{\mathcal{W}} = \big\{\langle b,c\rangle\big\}$. Observe that $Diff\,(\mathcal{W},\mathcal{I}) = \{A,r,s\}$, $Diff\,(\mathcal{W},\mathcal{I}') = \{A,r,s\}$, and $Dist\,(\mathcal{T},\mathcal{I}) = Dist\,(\mathcal{T},\mathcal{I}') = 3$.

Condition (D1) sets the interpretations $\mathcal{I}$, $\mathcal{I}'$ equally plausible (with respect to $\mathcal{T}$), whereas, condition (D2) demands $\mathcal{I}$ to be at least as plausible as $\mathcal{I}'$ (with respect to $\mathcal{T}$).

Evidently, condition (D2), like condition (D1), characterizes a particular class of DL revision operators.

8.4. **Parametrized Hamming-Based DL Revision Operators.** For specifying parametrized forms of Hamming-based DL revision operators, in the spirit of PD operators, an ordering over the elementary components of the underlying language is, first, required. Given that the "building blocks" of a DL language are the concept and role names, we shall consider (à la Peppas and Williams) a total preorder $\trianglelefteq$ over the set of all concept and role names $\mathcal{N}$.[20] By treating each concept and role name as a propositional variable, the preorder $\trianglelefteq$ can, straightforwardly, be extended to *sets* of concept and role names via Definition 5 (Section 5).

Consider that the difference *Diff* between interpretations is defined via Definition 16. Then, we can define a parametrized Hamming-based DL revision operator as a DL revision operator induced by means of (F∘), from the following preorders over interpretations (one for each TBox $\mathcal{T}$):

(PD1) $\mathcal{I} \preceq_{\mathcal{T}}^{\trianglelefteq} \mathcal{I}'$ iff there is a $\mathcal{W} \in [\mathcal{T}]$, such that, for all $\mathcal{W}' \in [\mathcal{T}]$, $\mathit{Diff}\,(\mathcal{W},\mathcal{I}) \trianglelefteq \mathit{Diff}\,(\mathcal{W}',\mathcal{I}')$.

Notice the resemblance of condition (PD1) with condition (PD) of Section 5, used in the case of propositional logic.

Given that concept and role names are treated as propositional variables, with an analogous line of reasoning as that in the proof of Theorem 4 of [26, p. 409], we derive that the preorder $\preceq_{\mathcal{T}}^{\trianglelefteq}$ is a total preorder, faithful to $\mathcal{T}$. Clearly then, we obtain immediately the following result.

Proposition 3. *The parametrized Hamming-based DL revision operators induced by means of condition (PD1), via (F∘), satisfy postulates (G1)–(G6).*

In the remainder of this section, we shall, also, introduce another type of parametrized Hamming-based DL revision operator, on top of a *refined* version of *Diff* of Definition 18.

Definition 20 (Refined Difference between Interpretations)**.** Let $\trianglelefteq$ be a total preorder over the set of concept and role names $\mathcal{N}$, and let $\mathcal{I}$, $\mathcal{I}'$ be any two interpretations of Ω. Moreover, let R be a unary function from $\mathcal{N}$ to the set of non-negative integers, used to quantitatively represent the preorder $\trianglelefteq$, for which $A \trianglelefteq B$ iff $R(A) \leqslant R(B)$, for any $A, B \in \mathcal{N}$. Then, the $\trianglelefteq$-refined difference between $\mathcal{I}$ and $\mathcal{I}'$, denoted by $\mathit{Diff}^{\,\trianglelefteq}(\mathcal{I},\mathcal{I}')$, is as follows:

$$\mathit{Diff}^{\,\trianglelefteq}(\mathcal{I},\mathcal{I}') = \sum_{N \in \mathcal{N}} \left| R(N) \cdot \mathit{Diff}_N(\mathcal{I},\mathcal{I}') \right|.$$

The above definition of difference can be applied to the notion of distance of Definition 19, in order to define a $\trianglelefteq$-*refined* distance between a TBox $\mathcal{T}$ and an interpretation $\mathcal{I}$, denoted by $\mathit{Dist}^{\,\trianglelefteq}(\mathcal{T},\mathcal{I})$. Based on this refined distance, consider the following two constraints on faithful preorders over interpretations.

(PD2) If $\mathit{Dist}\,(\mathcal{T},\mathcal{I}) < \mathit{Dist}\,(\mathcal{T},\mathcal{I}')$, then $\mathcal{I} \prec_{\mathcal{T}} \mathcal{I}'$.

(PD3) If $\mathit{Dist}\,(\mathcal{T},\mathcal{I}) = \mathit{Dist}\,(\mathcal{T},\mathcal{I}')$, then $\mathcal{I} \preceq_{\mathcal{T}} \mathcal{I}'$ iff $\mathit{Dist}^{\,\trianglelefteq}(\mathcal{T},\mathcal{I}) \leqslant \mathit{Dist}^{\,\trianglelefteq}(\mathcal{T},\mathcal{I}')$.

Conditions (PD2)–(PD3), together, ensure a preorder over interpretations analogous to a PD preorder in the propositional setting. According to (PD2)–(PD3), the ranking of interpretations takes place in two stages; (PD2) undertakes the first stage, whereas, (PD3) undertakes the second stage. In particular, condition (PD2) states that an interpretation $\mathcal{I}$ that is strictly closer to $\mathcal{T}$ than an interpretation $\mathcal{I}'$ (i.e., $\mathit{Dist}\,(\mathcal{T},\mathcal{I}) < \mathit{Dist}\,(\mathcal{T},\mathcal{I}')$) ought to be strictly more plausible than $\mathcal{I}'$, with respect to $\mathcal{T}$; this is a classical Hamming-based approach. Condition

[20]For ease of exposition, the same symbol (i.e., $\trianglelefteq$) for preorders over concept/role names and preorders over atoms is used.

(PD3), on the other hand, handles any two interpretations $\mathcal{I}$, $\mathcal{I}'$ that are equidistant from $\mathcal{T}$ (i.e., $\mathit{Dist}\,(\mathcal{T},\mathcal{I}) = \mathit{Dist}\,(\mathcal{T},\mathcal{I}')$), and orders them according to their $\trianglelefteq$-refined distance from $\mathcal{T}$.

As conditions (PD2)–(PD3) are constraints on faithful preorders over interpretations, they identify a certain class of parametrized Hamming-based DL revision operators that satisfy postulates (G1)–(G6).

We close this section noting that an interesting avenue for future research would be the definition of DL revision operators based on non-Hamming string metrics for the (dis)similarity of interpretations; the interested reader is referred to [21] for an indicative overview of such measures.

9. Conclusion

PD operators constitute a proper subclass of concrete AGM revision functions, induced by a total preorder over the set of propositional variables, with a plethora of favourable characteristics. In this article, the original PD-revision framework was strengthened with respect to the following three aspects: Firstly, a dynamic form of PD revision was defined, letting a changeable ranking over atoms. Following that, we showed that PD operators are not compatible with Horn revision. Accordingly, a thorough investigation of the relation between PD and Horn revision was conducted, which revealed an indirect interesting connection. Lastly, as the original definition of PD operators was formulated in classical propositional logic, a study on the adaptation of PD revision in the realm of DL was conducted, resulting in several Hamming-based DL revision operators, as well as various atom-parametrized forms of them.

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