Types of Rational Horn Revision Operators

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Abstract

In this article, we identify some interesting types of rational revision operators that implement Horn revision. In particular, we first define (both axiomatically and semantically) a class of Horn revision operators based on proper set inclusion of the atoms satisfied by possible worlds. Furthermore, we show that a well-behaved type of rational revision, called uniform revision, is Horn-compliant. This is demonstrated by proving that concrete Horn revision operators implement particular uniform-revision policies.

1 Introduction

Belief revision (or revision) is the process by which a rational agent changes their beliefs, in the light of new information [7]. A prominent approach that formalizes belief revision is that proposed by Alchourrón, Gärdenfors and Makinson in [1], now known as the *AGM paradigm*. Within the AGM paradigm, the agent's belief corpus is modelled by a logical theory K, also referred to as a *belief set*, new information (alias, *epistemic input*) is represented as a logical sentence φ, and the revision of K by φ is modelled as a (revision) function $*$ that maps K and φ to the revised (new) theory $K * \varphi$. Eight postulates, called the *AGM postulates for revision*, *axiomatically* characterize any *rational* revision operator, named *AGM revision function*. It has been, also, proven that any AGM revision function can be *semantically constructed* (specified) by means of a special kind of total preorders over possible worlds, called *faithful preorders* [8].

Given the nice properties of *Horn logic* (i.e., the Horn fragment of propositional logic), the AGM paradigm was modified by Delgrande and Peppas, so that it characterizes the class of AGM revision functions that map a Horn belief set and a Horn sentence to a (new) revised Horn belief set [5]; we shall refer to such AGM revision functions as *Horn AGM revision functions*.

In this article, we first identify (both *axiomatically* and *semantically*) an interesting *proper sub-class* of the class of Horn AGM revision functions, which is based on *proper set inclusion* of the atomic propositions satisfied by possible worlds (Section 4). Furthermore, we prove that a special type of well-behaved AGM revision functions, called *uniform-revision* (UR) operators —introduced in [3] and subsequently studied in detail in [2]— is Horn-compliant, in the sense that the class of UR operators *intersects* the class of Horn AGM revision functions (Section 5). This is demonstrated by proving that a concrete "off-the-shelf" Horn AGM revision function, proposed in [5], as well as some of the aforementioned inclusion-based Horn AGM revision functions are, as a matter of fact, particular UR operators.

2 Basic Notations and Conventions

We shall be working with a propositional language $\mathcal{L}$, built over a *finite*, non-empty set $\mathcal{P}$ of atoms, using the standard Boolean connectives, and governed by *classical propositional logic*. For a set of sentences Γ of $\mathcal{L}$, $Cn(\Gamma)$ denotes the set of all logical consequences of Γ; i.e.,

$Cn(\Gamma) = \{\varphi \in \mathcal{L} : \Gamma \models \varphi\}$. A *theory*, also referred to as a *belief set*, K is any deductively closed set of sentences of $\mathcal{L}$; i.e., $K = Cn(K)$. The set of all theories is denoted by $\mathbb{K}$.

A *literal* is an atom $p \in \mathcal{P}$ or its negation. For a set of literals Q, $|Q|$ denotes the cardinality of Q, and $\overline{Q}$ denotes the set of negated elements of Q; i.e., $\overline{Q} = \{\neg l : l \in Q\}$. A *possible world* (or, simply, *world*) r is any consistent set of literals, such that, for any atom $p \in \mathcal{P}$, either $p \in r$ or $\neg p \in r$. The set of all possible worlds is denoted by $\mathbb{M}$. The set of all atoms satisfied by a world $r \in \mathbb{M}$ is denoted by r^+; i.e., $r^+ = r \cap \mathcal{P}$. For a sentence (set of sentences) φ of $\mathcal{L}$, $[\varphi]$ is the set of worlds at which φ is true. Possible worlds will, occasionally, be represented as sequences of literals, and the negation of an atom p will be represented as $\overline{p}$.

A clause (i.e., a disjunction of literals) is called a *Horn clause* iff it contains *at most* one atom; e.g., $a \vee \neg b \vee \neg c$, where a, b, c are atoms. A *Horn formula* is a conjunction of Horn clauses. The *Horn language* $\mathcal{L}_H$ is the maximal subset of $\mathcal{L}$ containing only Horn formulas. The set of all Horn theories is denoted by $\mathbb{H}$. The Horn logic generated from $\mathcal{L}_H$ is specified by the consequence operator Cn_H, such that, for any set Γ of Horn formulas, $Cn_H(\Gamma) = Cn(\Gamma) \cap \mathcal{L}_H$. The *atoms-intersection* of two worlds $r, r' \in \mathbb{M}$ is a world denoted by $r \cap^+ r'$, and defined as follows: $r \cap^+ r' = (r^+ \cap r'^+) \cup \big(\overline{\mathcal{P} - (r^+ \cap r'^+)}\big)$.[1] An arbitrary formula (or theory) φ is Horn (i.e., $\varphi \in \mathcal{L}_H$) iff $r, r' \in [\varphi]$ entails $r \cap^+ r' \in [\varphi]$.

A *preorder* $\preceq$ over a set V is called *total* iff, for all $r, r' \in V$, $r \preceq r'$ or $r' \preceq r$. The strict part of $\preceq$ is denoted by $\prec$; i.e., $r \prec r'$ iff $r \preceq r'$ and $r' \not\preceq r$. The symmetric part of $\preceq$ is denoted by $\approx$; i.e., $r \approx r'$ iff $r \preceq r'$ and $r' \preceq r$. Also, $min(V, \preceq)$ denotes the set of all $\preceq$-minimal elements of V; i.e., $min(V, \preceq) = \Big\{r \in V : \text{for all } r' \in V, \text{ if } r' \preceq r, \text{ then } r \preceq r'\Big\}$.

3 AGM-Style Horn Revision

Within the AGM paradigm, the revision-process is modelled as a (binary) function $*$ that maps a theory K and a sentence φ to the revised (new) theory $K * \varphi$; i.e., $* : \mathbb{K} \times \mathcal{L} \mapsto \mathbb{K}$. The *AGM postulates for revision* —which are not presented herein due to space limitations— *axiomatically* characterize all *rational* revision functions, the so-called *AGM revision functions*.[2] Katsuno and Mendelzon proved that any AGM revision function can be *semantically* specified with the use of a special type of total preorders over all possible worlds, called *faithful preorders* [8].

Definition 1 (Faithful Preorder, [8]). *A total preorder $\preceq_K$ over $\mathbb{M}$ is faithful to a theory K iff the $\preceq_K$-minimal worlds are those satisfying K; i.e., $min(\mathbb{M}, \preceq_K) = [K]$.*

Intuitively, $r \preceq_K r'$ holds whenever r is at least as *plausible* (relative to K) as r'.

Theorem 1 ([8]). *For every theory K and any sentence φ of $\mathcal{L}$, an AGM revision function $*$ can be defined (specified) by means of the following condition:*

$$\textbf{(F*)} \qquad [K * \varphi] = min([\varphi], \preceq_K).$$

3.1 The AGM Paradigm in the Horn Setting

Since satisfiability in the Horn setting —i.e., evaluating whether $\varphi \in H$ is true, where $H \in \mathbb{H}$ and $\varphi \in \mathcal{L}_H$— can be determined in *linear time* [6, 9], *Horn revision* has gained great interest. A notable approach, in this regard, constitutes the work of Delgrande and Peppas [5]. In that work, the authors *axiomatically* characterized the class of Horn AGM revision functions, by

[1] For instance, $\{a, b, c\} \cap^+ \{a, \neg b, \neg c\} = \{a, \neg b, \neg c\}$ and $\{\neg a, b, c\} \cap^+ \{a, \neg b, \neg c\} = \{\neg a, \neg b, \neg c\}$.

[2] See [7, Section 3.3] or [10, Section 8.3.1] for a detailed presentation of the AGM postulates for revision.

strictly strengthening the class of AGM revision functions, with the aid of an extra postulate that supplements the AGM postulates for revision — for details, the reader is referred to [5]. We recall that a Horn AGM revision function $*$ is an AGM revision function that maps a Horn theory and a Horn formula to a (new) revised Horn theory; i.e., $* : \mathbb{H} \times \mathcal{L}_H \mapsto \mathbb{H}$.

Now, consider the following constraint on a total preorder $\preceq_K$ over $\mathbb{M}$, which is faithful to a theory K, introduced by Zhuang and Pagnucco in [11].

(H) If $r \approx_K r'$, then $r \cap^+ r' \preceq_K r$.

Condition (H) says that whenever two worlds r and r' are equidistant from the beginning of a preorder $\preceq_K$, then the world $r \cap^+ r'$, resulting from their atoms-intersection, cannot appear later in $\preceq_K$. The results of Delgrande and Peppas [5], along with important results established by Zhuang and Pagnucco in [11], entail the following *representation theorem*.

Theorem 2 ([11, 5]). *Let $*$ be an AGM revision function, and let $\{\preceq_K\}_{K \in \mathbb{K}}$ be the family of total preorders over worlds that correspond to $*$, by means of condition (F$*$). Then, $*$ is a Horn AGM revision function iff $\{\preceq_K\}_{K \in \mathbb{K}}$ satisfies condition (H).*

3.2 Basic Horn Revision

Delgrande and Peppas not only axiomatically characterized AGM-style Horn revision, but also proposed some interesting concrete Horn AGM revision functions [5]. In this subsection, we present one of their proposals, which is inspired by the *Hamming-based* Dalal's approach [4].

The *basic Horn revision function*, denoted by $\diamond$, is defined as the (*unique*) Horn AGM revision function induced, via condition (F$*$), from a family $\{\preceq_H\}_{H \in \mathbb{H}}$ of total preorders over $\mathbb{M}$, that satisfies the following constraint.

(BH) $r \preceq_H r'$ iff $|r^+| \leqslant |r'^+|$.

Condition (BH) orders the relative plausibility of worlds according to the number of atoms they satisfy; notice that (BH) *uniquely* specifies $\preceq_H$, thus, $\diamond$ is unique.

4 Inclusion-Based Horn Revision

In this section, we introduce a new *proper sub-class* of Horn AGM revision functions, based on *proper set inclusion* of atoms of worlds. First, we introduce the next definition.

Definition 2 (Atoms-Ordered Theory). *Let K be a consistent theory of $\mathcal{L}$. We shall say that K is atoms-ordered iff the atoms of the worlds of $[K]$ are totally ordered with respect to proper set inclusion; i.e., for any two worlds $r, r' \in [K]$, either $r^+ \subset r'^+$ or $r'^+ \subset r^+$.*

On that premise, consider the following postulate (PI). The semantic condition that corresponds to (PI) is condition (PIS), also presented below, which constrains a total preorder $\preceq_K$ over $\mathbb{M}$, which is faithful to a theory K.

(PI) For any consistent φ of $\mathcal{L}$, $K * \varphi$ is atoms-ordered.

(PIS) If $r \approx_K r'$, then either $r^+ \subset r'^+$ or $r'^+ \subset r^+$.

According to (PIS), the faithful preorder $\preceq_K$ is defined so that the atoms of the non-K-worlds of every equivalent class (layer) of $\preceq_K$ are totally ordered with respect to *proper set inclusion*. An example of a faithful preorder that respects condition (PIS) is shown below, for $\mathcal{P} = \{a, b, c\}$ and the Horn belief set $H = Cn_H\big((\neg a \vee b) \wedge \neg c\big)$ — notice that $[H] = \big\{\{a, b, \neg c\}, \{\neg a, b, \neg c\}, \{\neg a, \neg b, \neg c\}\big\}$.

$$\begin{array}{ccccc} ab\overline{c} & & & & abc \\ \overline{a}b\overline{c} & \prec_H & \begin{array}{c}\overline{a}bc \\ \overline{a}\overline{b}c\end{array} & \prec_H & a\overline{b}c \\ \overline{a}\overline{b}\overline{c} & & & & a\overline{b}\overline{c} \end{array}$$

Theorem 3 establishes the connection between postulate (PI) and constraint (PIS).

Theorem 3. *Let $*$ be an AGM revision function, and let $\{\preceq_K\}_{K\in\mathbb{K}}$ be the family of total preorders over $\mathbb{M}$ that correspond to $*$, by means of condition (F$*$). Then, $*$ satisfies postulate (PI) iff $\{\preceq_K\}_{K\in\mathbb{K}}$ satisfies condition (PIS).*

Proof. For the left-to-right implication, assume that $*$ satisfies (PI). We show that $\{\preceq_K\}_{K\in\mathbb{K}}$ satisfies (PIS). Let r, r' be two worlds of $\mathbb{M}$, such that $r \approx_K r'$. Define φ to be a sentence of $\mathcal{L}$, such that $[\varphi] = \{r, r'\}$. Then, condition (F$*$) entails that $[K * \varphi] = \{r, r'\}$. Therefore, from condition (PI), we have that either $r^+ \subset r'^+$ or $r'^+ \subset r^+$, as desired.

For the right-to-left implication, assume that $\{\preceq_K\}_{K\in\mathbb{K}}$ satisfies (PIS). We show that $*$ satisfies (PI). For any consistent sentence φ of $\mathcal{L}$, it follows, from condition (F$*$), that all worlds in $[K * \varphi]$ are equally plausible, with respect to K. That is, for any two worlds $r, r' \in [K * \varphi]$, it is true that $r \approx_K r'$. Hence, from (PIS), we have that either $r^+ \subset r'^+$ or $r'^+ \subset r^+$. This again entails that $K * \varphi$ is atoms-ordered, as desired. ■

The next theorem shows that any AGM revision function that respects postulate (PI) is a Horn AGM revision function.

Theorem 4. *Let $*$ be an AGM revision function. If $*$ satisfies postulate (PI), then $*$ is a Horn AGM revision function.*

Proof. Let H be a Horn belief set, and let $\preceq_H$ be the faithful preorder that $*$ assigns at H, via (F$*$). Since $*$ satisfies (PI), $\preceq_H$ satisfies (PIS). It suffices to show that $\preceq_H$ satisfies condition (H). Let r, r' be two worlds of $\mathbb{M}$, such that $r \approx_H r'$. We will show that $r \cap^+ r' \preceq_H r$. If $r \cap^+ r' \in [H]$, this is clearly true. Assume, therefore, that $r \cap^+ r' \notin [H]$. Then, since H is a Horn theory, not both r and r' can be members of $[H]$ (for otherwise $r \cap^+ r'$ would, also, belong to $[H]$). Since one of r, r' is not in $[H]$, from $r \approx_H r'$, we derive that neither of r, r' belong to $[H]$. From $r \approx_H r'$, (PIS) entails that either $r^+ \subset r'^+$ or $r'^+ \subset r^+$. This again implies that $r \cap^+ r' \approx_H r \approx_H r'$; that is, $r \cap^+ r' \preceq_H r$, as desired. ■

Theorem 4, along with Theorem 5 shown below, prove that the family of Horn AGM revision functions identified by postulate (PI) constitutes a *proper* sub-class of the whole class of Horn AGM revision functions.

Theorem 5. *There exists a Horn AGM revision function that does not satisfy postulate (PI).*

Proof. Let $\mathcal{P} = \{a, b, c\}$, and let H be a Horn belief set such that $H = Cn_H\big((\neg a \vee b) \wedge \neg c\big)$. Clearly, $[H] = \big\{\{a, b, \neg c\}, \{\neg a, b, \neg c\}, \{\neg a, \neg b, \neg c\}\big\}$. Let $*$ be a Horn AGM revision function that assigns at H (via (F$*$)) the next total preorder $\preceq_H$ over $\mathbb{M}$, which respects condition (H).

$$\begin{array}{c} ab\overline{c} \\ \overline{a}b\overline{c} \\ \overline{a}\overline{b}\overline{c} \end{array} \quad \prec_H \quad \begin{array}{c} \overline{a}bc \\ \overline{a}\overline{b}c \\ a\overline{b}c \end{array} \quad \prec_H \quad \begin{array}{c} abc \\ a\overline{b}\overline{c} \end{array}$$

Observe that $\preceq_H$ does *not* respect condition (PIS), hence, $*$ does not satisfy postulate (PI). ■

5 Uniform Revision is Horn-Compliant

This section investigates *uniform revision* in the realm of Horn logic. A uniform-revision (UR) operator is *uniquely* specified by means of a *single* total preorder $\preccurlyeq$ over all possible worlds of $\mathbb{M}$, which essentially expresses their *prior* relative plausibility, as considered by a particular rational agent. This is accomplished since a total preorder $\preccurlyeq$ suffices to *uniquely* specify the agent's revision policy, with respect to *every* belief set of the language (by means of condition (URS1), presented subsequently) [3, 2].

It proves to be the case that a UR operator is any AGM revision function $*$ satisfying the following postulate (UR); the semantic condition that corresponds to (UR) is condition (URS1), which is, in turn, equivalent to condition (URS2) (where $K, T \in \mathbb{K}$) [3, 2].

(UR) For any $\neg\varphi \in K \cap T$, $K * \varphi = T * \varphi$.

(URS1) For any $r, r' \notin [K]$, $r \preceq_K r'$ iff $r \preccurlyeq r'$.

(URS2) For any $r, r' \notin [K] \cup [T]$, $r \preceq_K r'$ iff $r \preceq_T r'$.

Against this background, Theorem 6 proves that basic Horn revision encodes, as a matter of fact, a particular uniform-revision policy.

Theorem 6. *The basic Horn revision function $\diamond$ is a UR operator.*

Proof. Let H, H' be any two Horn belief sets, and let $\preceq_H$, $\preceq_{H'}$ be the faithful preorders that $\diamond$ assigns (via (F$*$)) at H, H', respectively. It suffices to show that $\preceq_H$, $\preceq_{H'}$ satisfy condition (URS2). Since $\preceq_H$, $\preceq_{H'}$ satisfy condition (BH), it is true that, for any worlds $z, z' \notin [H]$ and any worlds $u, u' \notin [H']$, $z \preceq_H z'$ iff $|z^+| \leqslant |z'^+|$, and $u \preceq_{H'} u'$ iff $|u^+| \leqslant |u'^+|$. This again entails that, for any worlds $r, r' \notin [H] \cup [H']$, $r \preceq_H r'$ iff $r \preceq_{H'} r'$. Therefore, the faithful preorders $\preceq_H$, $\preceq_{H'}$ satisfy condition (URS2), as desired. ■

The aforementioned theorem, essentially, shows that the class of UR operators *intersects* the class of Horn AGM revision functions. Therefore, uniform revision is Horn-compliant — this is an important result that comes to extend the favourable properties of uniform revision.

Remark 1. *One can easily find UR operators that are not Horn AGM revision functions, as well as Horn AGM revision functions that are not UR operators.*

Next, Theorem 7 proves that there exist uniform-revision policies that implement inclusion-based Horn revision.

Theorem 7. *There exists a UR operator that satisfies postulate (PI).*

Proof. Let $\preccurlyeq$ be a total preorder over $\mathbb{M}$, defined as follows ($\prec\!\!\cdot$ denotes the strict part of $\preccurlyeq$):

$$\begin{array}{c} ab\overline{c} \\ \overline{a}b\overline{c} \\ \overline{a}\overline{b}\overline{c} \end{array} \quad \prec\!\!\cdot \quad \begin{array}{c} \overline{a}bc \\ \overline{a}\overline{b}c \end{array} \quad \prec\!\!\cdot \quad \begin{array}{c} abc \\ a\overline{b}c \\ a\overline{b}\overline{c} \end{array}$$

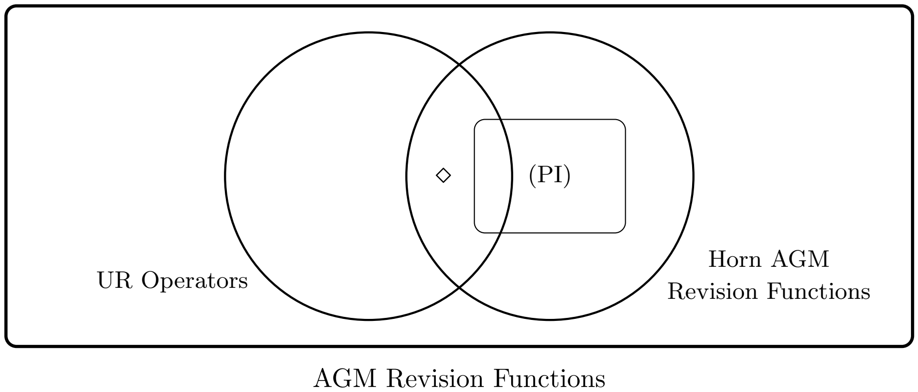

Figure 1: The types of AGM revision functions discussed herein.

The preorder $\preccurlyeq$ specifies (via (URS1)) a unique family $\{\preceq_K\}_{K\in\mathbb{K}}$ of total preorders over $\mathbb{M}$, which, in turn, induces (via (F$*$)) a unique UR operator $*$. Since $\preccurlyeq$ respects condition (PIS), it follows that $\{\preceq_K\}_{K\in\mathbb{K}}$ respects (PIS) as well. Hence, $*$ satisfies postulate (PI), as desired. ■

Remark 2. *One can easily find inclusion-based Horn AGM revision functions that are not UR operators. Moreover, there exists a Horn AGM revision function, which is, also, a UR operator —namely, the basic Horn revision function— that can be verified to violate postulate (PI).*

It can be shown that any class of AGM revision functions that are induced from faithful preorders which specify the relative plausibility of possible worlds *regardless* of the (information contained in the) respective belief set —such as the basic Horn revision function $\diamond$ and the inclusion-based Horn AGM revision functions, considered in this section— *intersects* the class of UR operators; the details are left for future research.

The results of the present work are summarized in Figure 1, which depicts the types of AGM revision functions discussed herein.

6 Conclusion

In this work, we identified some interesting types of Horn AGM revision functions. In particular, we defined (axiomatically and semantically) a proper sub-class of Horn AGM revision functions, based on proper set inclusion of the atoms of possible worlds. We, also, showed that the well-behaved uniform revision is Horn-compliant, since concrete Horn AGM revision functions are, in fact, particular UR operators. Given the critical role that Horn logic plays in belief revision, further research on other solid types of Horn AGM revision functions is quite compelling.

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