TOPOLOGICAL RADICALS OF SEMICROSSED PRODUCTS

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ABSTRACT. We characterize the hypocompact radical of a semicrossed product in terms of properties of the dynamical system. We show that an element A of a semicrossed product is in the hypocompact radical if and only if the Fourier coefficients of A vanish on the closure of the recurrent points and the 0-Fourier coefficient vanishes also on the largest perfect subset of X.

1. Introduction and Preliminaries

Let $\mathcal{B}$ be a Banach algebra. An element a of $\mathcal{B}$ is said to be compact if the map $M_{a,a} : \mathcal{B} \to \mathcal{B}$, $x \mapsto axa$ is compact. Following Shulman and Turovskii [17, 3.2] we will call a Banach algebra $\mathcal{B}$ *hypocompact* if any nonzero quotient $\mathcal{B}/\mathcal{J}$ by a closed ideal $\mathcal{J}$ contains a nonzero compact element. We will say that an ideal $\mathcal{J}$ of a Banach algebra $\mathcal{B}$ is hypocompact if it is hypocompact as an algebra. Shulman and Turovskii have proved that any Banach algebra $\mathcal{B}$ has a largest hypocompact ideal [17, Corollary 3.10]. This ideal is closed and is called the hypocompact radical of $\mathcal{B}$. We will denote it by $\mathcal{B}_{hc}$.

The hypocompact radical of Banach algebras was studied within the framework of the theory of topological radicals [17, 18]. This theory originated with Dixon [6] and was further developed by Shulman and Turovskii in a series of papers [13, 14, 15, 17, 18] and by Kissin, Shulman and Turovskii [16]. The theory of topological radicals has applications to various problems of Operator Theory and Banach algebras.

It follows from [4, Lemma 8.2], that the hypocompact radical contains the ideal generated by the compact elements. If $\mathcal{X}$ is a Banach space, we shall denote by $\mathcal{B}(\mathcal{X})$ the Banach algebra of all bounded linear operators on $\mathcal{X}$ and by $\mathcal{K}(\mathcal{X})$ the Banach subalgebra of all compact operators on $\mathcal{X}$. Vala has shown in [19] that an element $a \in \mathcal{B}(\mathcal{X})$ is a compact element if and only if $a \in \mathcal{K}(\mathcal{X})$. It follows that if $\mathcal{H}$ is a separable Hilbert space, the hypocompact radical of $\mathcal{B}(\mathcal{H})$ is $\mathcal{K}(\mathcal{H})$. Indeed, the ideal $\mathcal{K}(\mathcal{H})$ is the only proper ideal of $\mathcal{B}(\mathcal{H})$ while the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ does not have any non-zero compact element [8, section 5].

Shulman and Turovskii observe in [17, p. 298] that there exist Banach spaces $\mathcal{X}$, such that the hypocompact radical $\mathcal{B}(\mathcal{X})_{hc}$ of $\mathcal{B}(\mathcal{X})$ contains all the weakly compact operators and contains strictly the ideal of compact operators $\mathcal{K}(\mathcal{X})$.

Argyros and Haydon constructed in [3] a Banach space $\mathcal{X}$ such that every operator in $\mathcal{B}(\mathcal{X})$ is a scalar multiple of the identity plus a compact operator. It follows that $\mathcal{B}(\mathcal{X})/\mathcal{K}(\mathcal{X})$ is finite-dimensional and hence the hypocompact radical of $\mathcal{B}(\mathcal{X})$ coincides with $\mathcal{B}(\mathcal{X})$.

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A nest $\mathcal{N}$ on a Hilbert space $\mathcal{H}$ is a totally ordered family of closed subspaces of $\mathcal{H}$ containing $\{0\}$ and $\mathcal{H}$, which is closed under intersection and closed span. If $\mathcal{N}$ is a nest on a Hilbert space $\mathcal{H}$, the nest algebra associated to $\mathcal{N}$ is the (non selfadjoint) algebra of all operators $T \in \mathcal{B}(\mathcal{H})$ which leave each member of $\mathcal{N}$ invariant. The hypocompact radical of a nest algebra was characterized in [1].

We recall the construction of the semicrossed product we will consider in this work. Let X be a locally compact metrizable space and $\phi : X \to X$ a homeomorphism. The pair (X, ϕ) is called a dynamical system. An action of $\mathbb{Z}_+$ on $C_0(X)$ by isometric $*$-automorphisms α_n, $n \in \mathbb{Z}_+$ is obtained by defining $\alpha_n(f) = f \circ \phi^n$. We write the elements of the Banach space $\ell^1(\mathbb{Z}_+, C_0(X))$ as formal series $A = \sum_{n\in\mathbb{Z}_+} U^n f_n$ with the norm given by $\|A\|_1 = \sum_{n\in\mathbb{Z}_+} \|f_n\|_{C_0(X)}$. The multiplication on $\ell^1(\mathbb{Z}_+, C_0(X))$ is defined by setting

$$U^n f U^m g = U^{n+m}(\alpha^m(f)g)$$

and extending by linearity and continuity. With this multiplication, $\ell^1(\mathbb{Z}_+, C_0(X))$ is a Banach algebra.

The Banach algebra $\ell^1(\mathbb{Z}_+, C_0(X))$ can be faithfully represented as a (concrete) operator algebra on a Hilbert space. This is achieved by assuming a faithful action of $C_0(X)$ on a Hilbert space $\mathcal{H}_0$. Then, we can define a faithful contractive representation π of $\ell_1(\mathbb{Z}_+, C_0(X))$ on the Hilbert space $\mathcal{H} = \mathcal{H}_0 \otimes \ell^2(\mathbb{Z}_+)$ by defining $\pi(U^n f)$ as

$$\pi(U^n f)(\xi \otimes e_k) = \alpha^k(f)\xi \otimes e_{k+n}.$$

The *semicrossed product* $C_0(X) \times_\phi \mathbb{Z}_+$ is the closure of the image of $\ell^1(\mathbb{Z}_+, C_0(X))$ in $\mathcal{B}(\mathcal{H})$ in the representation just defined, where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$. Note that the semicrossed product is in fact independent of the faithful action of $C_0(X)$ on $\mathcal{H}_0$ (up to isometric isomorphism) [7]. We will denote the semicrossed product $C_0(X) \times_\phi \mathbb{Z}_+$ by $\mathcal{A}$ and an element $\pi(U^n f)$ of $\mathcal{A}$ by $U^n f$ to simplify the notation. We refer to [12, 7, 5], for more information about the semicrossed product.

For $A = \sum_{n\in\mathbb{Z}_+} U^n f_n \in \ell^1(\mathbb{Z}_+, C_0(X))$, we call $f_n \equiv E_n(A)$ the nth *Fourier coefficient* of A. The maps $E_n : \ell^1(\mathbb{Z}_+, C_0(X)) \to C_0(X)$ are contractive in the (operator) norm of $\mathcal{A}$, and therefore they extend to contractions $E_n : \mathcal{A} \to C_0(X)$. An element A of the semicrossed product $\mathcal{A}$ is 0 if and only if $E_n(A) = 0$ for all $n \in \mathbb{Z}+$ and thus A is completely determined by its Fourier coefficients. We will denote A by the formal series $A = \sum_{n\in\mathbb{Z}_+} U^n f_n$, where $f_n = E_n(A)$. Note however that the series $\sum_{n\in\mathbb{Z}_+} U^n f_n$ does not in general converge to A [12, II.9, IV.2 Remark].

In this paper we characterize the hypocompact radical of a semicrossed product in terms of properties of the dynamical system. We show that an element A of a semicrossed product is in the hypocompact radical if and only if the Fourier coefficients of A vanish on the closure of the recurrent points and the 0-Fourier coefficient vanishes also on the largest perfect subset of X.

2. The hypocompact radical

To obtain the characterization of the hypocompact radical of a semicrossed product we recall the following properties of the hypocompact radical of a Banach algebra proved by Shulman and Turovskii in [17].

Theorem 2.1. *Let $\mathcal{B}$ be a Banach algebra and $\mathcal{I}$ a closed ideal of $\mathcal{B}$.*

(1) *If $\mathcal{B}$ is hypocompact, then $\mathcal{I}$ and $\mathcal{B}/\mathcal{I}$ are hypocompact* [17, Corollary 3.9].
(2) *If $\mathcal{I}$ and $\mathcal{B}/\mathcal{I}$ are hypocompact, then $\mathcal{B}$ is hypocompact* [17, Corollary 3.9].
(3) *Let $p : \mathcal{B} \to \mathcal{B}/\mathcal{I}$ be the quotient map. Then $p(\mathcal{B}_{hc}) \subseteq (\mathcal{B}/\mathcal{I})_{hc}$*[17, Corollary 3.13].

Let X be a locally compact metrizable space. We shall use the characterization of the hypocompact radical of $C_0(X)$ which may be obtained using [18, Corollary 8.19 & Theorem 8.22]. We provide a proof for completeness.

A point $x \in X$ is called *accumulation point* of X, if $x \in \overline{X \setminus \{x\}}$. The set of the accumulation points of X is denoted X_a. If $x \in X \setminus X_a$, then the point x is called an *isolated point.* A subset Y of a topological space is said to be *dense in itself*, if it contains no isolated points. If Y is closed and dense in itself, it is said to be a *perfect set.* The set Y is said to be a *scattered set*, if it does not contain dense in themselves subsets.

It is well known that every space is the disjoint union of a perfect and a scattered one, and this decomposition is unique [9, Theorem 3, p 79]. If X is a locally compact metrizable space, we write $X = X_p \cup X_s$ where X_p is the perfect set and X_s is the scattered set.

Theorem 2.2. *If X is a locally compact metrizable space, then*

$$C_0(X)_{hc} = \{f \in C_0(X) : f(X_p) = \{0\}\}.$$

Proof. Let $\mathcal{I}$ be the ideal $\{f \in C_0(X) : f(X_p) = \{0\}\}$ of $C_0(X)$. The ideal $\mathcal{I}$ is isomorphic to $C_0(X_s)$. We show that every non-zero quotient of $\mathcal{I}$ by a closed ideal has a non-zero compact element. Let $\mathcal{J}$ be a closed ideal of $\mathcal{I}$ and S a closed subset of X_s such that $\mathcal{J} = \{f \in C_0(X_s) : f(S) = \{0\}\}$. The quotient algebra $\mathcal{I}/\mathcal{J}$ is isomorphic to $C_0(S)$. Hence it suffices to prove that the algebra $C_0(S)$ has a non-zero compact element. Since the set S is contained in X_s it is scattered, and it contains an isolated point y. Let $\chi_{\{y\}}$ be the characteristic function of $\{y\}$. Then, the operator $M_{\chi_{\{y\}},\chi_{\{y\}}} : C_0(S) \to C_0(S)$ is a rank-one operator and hence, $\chi_{\{y\}}$ is a compact element of the algebra $C_0(S)$. It follows that $\mathcal{I} \subseteq C_0(X)_{hc}$.

We show now that $\mathcal{I} = C_0(X)_{hc}$. Assuming that $\mathcal{I} \neq C_0(X)_{hc}$ we will prove that the quotient algebra $C_0(X)_{hc}/\mathcal{I}$ contains no non-zero compact elements. This implies that $\mathcal{I} = C_0(X)_{hc}$ by Theorem 2.1. Let $f \in C_0(X)_{hc} \setminus \mathcal{I}$. There exists $x_p \in X_p$, such that $f(x_p) \neq 0$ and an open neighborhood U_p of x_p, such that

$$|f(x)| > \frac{|f(x_p)|}{2}, \quad \forall x \in U_p.$$

Consider a sequence of points $\{x_i\}_{i\in\mathbb{N}} \subseteq U_p \cap X_p$ and a sequence of open subsets $\{V_i\}_{i\in\mathbb{N}}$ of X, such that $x_i \in V_i \subseteq U_p$ and $V_i \cap V_j = \emptyset$ for $i \neq j$.

By Urysohn's lemma there exists a sequence of functions $\{h_i\}_{i\in\mathbb{N}}$ such that $h_i(x_i) = 1$ and $h_i(X \setminus V_i) = \{0\}$. Let $q : C_0(X)_{hc} \to C_0(X)_{hc}/\mathcal{I}$ be the quotient map. We estimate for $i \neq j$:

$$\begin{aligned}\|M_{q(f),q(f)}(q(h_i)) - M_{q(f),q(f)}(q(h_j))\| &= \inf_{g\in\mathcal{I}} \|f^2h_i - f^2h_j + g\| \\ &\geq \inf_{g\in\mathcal{I}} |(f^2h_i - f^2h_j + g)(x_i)| \\ &= |f^2(x_i)| > \frac{|f(x_p)|^2}{4}.\end{aligned}$$

Hence, the sequence $\{M_{q(f),q(f)}(q(h_i))\}_{i\in\mathbb{N}}$ has no convergent subsequence, which implies that the element $q(f)$ is non compact. □

Recall that a set $Y \subseteq X$ is called wandering if the sets $\phi^{-1}(Y), \phi^{-2}(Y), ...$ are pairwise disjoint. Since ϕ is a homeomorphism, this condition is equivalent to the condition that $\phi^m(Y) \cap \phi^n(Y) = \emptyset$, for all $m, n \in \mathbb{Z}_+, m \neq n$. A point $x \in X$ is called wandering if it possesses an open wandering neighborhood. Otherwise it is called non wandering. We will denote by X_w the set of wandering points of X. It is clear that X_w is the the union of all open wandering subsets of X.

Let X_1 be the set of non wandering points of X and set $\phi_1 = \phi|_{X_1}$ the restriction of ϕ to X_1. We thus obtain a dynamical system (X_1, ϕ_1). Define by transfinite recursion a family (X_γ, ϕ_γ) of dynamical systems. If (X_γ, ϕ_γ) is defined, then set $X_{\gamma+1}$ the set of non wandering points of the dynamical system (X_γ, ϕ_γ) and $\phi_{\gamma+1} = \phi|X_{\gamma+1}$. If γ is a limit ordinal and the systems (X_β, ϕ_β) have been defined for all $\beta < \gamma$, set $X_\gamma = \cap_{\beta<\gamma} X_\beta$ and $\phi_\gamma = \phi|_{X_\gamma}$ the restriction of ϕ to X_γ. This process must stop at some ordinal γ_0, since the cardinality of the family cannot exceed the cardinality of the power set of X. The following is [7, Lemma 13].

Proposition 2.3. *The set X_{γ_0} is the closure of the set of recurrent points X_r of the system (X, ϕ).*

If γ is an ordinal $\gamma \leq \gamma_0$, we will denote by $\mathcal{I}_\gamma$ the ideal

$$\{A \in \mathcal{A} : E_0(A) = 0, E_n(A)(X_\gamma) = \{0\}, \forall n \in \mathbb{Z}_+, n \geq 1\}.$$

The proof of the following lemma is straightforward, and is omitted.

Lemma 2.4. *If γ is a limit ordinal, then $I_\gamma = \overline{\cup_{\beta<\gamma} I_\beta}$.*

It is known that the ideal generated by the compact elements of $\mathcal{A}$ is contained in the hypocompact radical [4]. We will need the following characterization of this ideal which is proved in [2].

Theorem 2.5. *The ideal generated by the compact elements of $\mathcal{A}$ is the set*

$$\{A \in \mathcal{A} \mid E_n(A)(X \setminus X_w) = \{0\},\ \forall n \in \mathbb{Z}_+ \text{ and } E_0(A)(X_a) = \{0\}\}.$$

The following is the main result of the paper.

Theorem 2.6. *The hypocompact radical $\mathcal{A}_{hc}$ of $\mathcal{A}$ is equal to*

$$\mathcal{I} = \{A \in \mathcal{A} : E_0(A)(X_p) = 0, E_n(A)(X_{\gamma_0}) = \{0\}, \forall n \in \mathbb{Z}_+\}.$$

Proof. **1st step**

We shall prove that $\mathcal{I}$ is contained in $\mathcal{A}_{hc}$. We first prove that $\mathcal{I}_{\gamma_0}$ is contained in $\mathcal{A}_{hc}$. Assume the contrary.

It follows from Theorem 2.5 that $\mathcal{I}_1$ is contained in the ideal generated by the compact elements. The hypocompact radical contains the ideal generated by the compact elements [4], and hence $\mathcal{I}_1$ is contained in $\mathcal{A}_{hc}$.

Let β be the least ordinal $\beta \leq \gamma_0$ such that $\mathcal{I}_\beta$ is not contained in $\mathcal{A}_{hc}$. We show that β is a successor. If not, since $\mathcal{I}_\gamma \subseteq \mathcal{A}_{hc}$ for all $\gamma < \beta$, we obtain from Lemma 2.4 that $\mathcal{I}_\beta = \overline{\cup_{\gamma<\beta} \mathcal{I}_\gamma} \subseteq \mathcal{A}_{hc}$, which is absurde. Hence, β is a successor.

We are going to prove that $\mathcal{I}_\beta$ is a hypocompact algebra. Consider the algebra $\mathcal{I}_\beta / \mathcal{I}_{\beta-1}$. It suffices to show that $\mathcal{I}_\beta / \mathcal{I}_{\beta-1}$ is hypocompact, since the class of hypocompact algebras is closed under extensions and the ideal $\mathcal{I}_{\beta-1}$ is hypocompact (Theorem 2.1).

We show that the algebra $\mathcal{I}_\beta/\mathcal{I}_{\beta-1}$ is generated by the compact elements it contains and hence is a hypocompact algebra by [4].

Let $A \in \mathcal{I}_\beta$. It follows from the condition defining $\mathcal{I}_\beta$, that $U^n E_n(A) \in \mathcal{I}_\beta$, for all $n \in \mathbb{Z}_+, n \geq 1$. Hence, it suffices to show that the image of $U^n E_n(A)$ under the natural map $\pi : \mathcal{I}_\beta \to \mathcal{I}_\beta/\mathcal{I}_{\beta-1}$ is contained in the ideal generated by the compact elements of $\mathcal{I}_\beta/\mathcal{I}_{\beta-1}$. It suffices to see this for an element of $\mathcal{I}_\beta$ of the form $U^n f$ with f compactly supported. It follows from [7, Lemma 14], that f can be written as a finite sum $f = \sum f_i$ where each f_i has compact support contained in an open set V_i such that $V_i \cap X_{\beta-1}$ is wandering for the system $(X_{\beta-1}, \phi_{\beta-1})$ and $U^n f_i \in \mathcal{I}_\beta$, for all i.

Hence, it suffices to prove that $\pi(U^n f)$ is a compact element, where f has compact support contained in an open set V, such that $V \cap X_{\beta-1}$ is wandering for the system $(X_{\beta-1}, \phi_{\beta-1})$.

We calculate:

$$U^n f(\sum U^m g_m)U^n f = \sum U^{2n+m} f \circ \phi^{m+n} g_m \circ \phi^n f,$$

for $\sum U^m g_m \in \mathcal{I}_\beta$.

Since $n \geq 1$, we have $n+m \geq 1$, for all $m \in \mathbb{Z}_+$, and consequently $f \circ \phi^{m+n} f = 0$ on $X_{\beta-1}$, for all $m \in \mathbb{Z}^+$ since $V \cap X_{\beta-1}$ is wandering. Hence, $U^n f(U^m g_m)U^n f = U^{2n+m} f \circ \phi^{m+n} g_m \circ \phi^n f \in \mathcal{I}_{\beta-1}$.

Thus, $\pi(U^n f)$ is a compact element of $\mathcal{I}_\beta/\mathcal{I}_{\beta-1}$, and $\mathcal{I}_\beta$ is a hypocompact ideal which is a contradiction. We conclude that $\mathcal{I}_{\gamma_0}$ is contained in $\mathcal{A}_{hc}$. Now, $\mathcal{I}/\mathcal{I}_{\gamma_0}$ is isomorphic to $\{f \in C_0(X) : f(X_p \cup X_{\gamma_0}) = \{0\}\}$ which is a hypocompact algebra by Theorem 2.2. It follows from Theorem 2.1 that $\mathcal{I}$ is a hypocompact ideal, and hence it is contained in $\mathcal{A}_{hc}$.

2nd step

We show now that $\mathcal{A}_{hc} = \mathcal{I}$. We will suppose that $\mathcal{I} \subsetneq \mathcal{A}_{hc}$ and we will prove that the quotient algebra $\mathcal{A}_{hc}/\mathcal{I}$, contains no non-zero compact elements. This implies that $\mathcal{A}_{hc} = \mathcal{I}$ by Theorem 2.1.

Let $A \in \mathcal{A}_{hc} \setminus \mathcal{I}$ and set $E_m(A) = f_m$, for all $m \in \mathbb{Z}_+$. Since the map E_0 is a continuous homomorphism from $\mathcal{A}$ onto $C_0(X)$, it follows from Theorem 2.1 that $E_0(\mathcal{A}_{hc}) \subseteq C_0(X)_{hc}$ and hence by Theorem 2.2 we have $E_0(A)(X_p) = \{0\}$.

Since $A \notin \mathcal{I}$, it follows from Proposition 2.3 that there exists $m \in \mathbb{Z}_+$ such that $f_m(X_r) \neq \{0\}$. We set

$$m_0 = \min\{m \in \mathbb{Z}_+ : f_m(X_r) \neq \{0\}\},$$

and we consider $x_0 \in X_r$ such that $f_{m_0}(x_0) \neq 0$. There exists an open neighborhood U_0 of x_0 such that

$$|f_{m_0}(x)| > \frac{|f_{m_0}(x_0)|}{2} \quad , \quad \forall x \in U_0. \tag{1}$$

Since x_0 is a recurrent point, there exist an open neighborhood V_0 of x_0 such that $\overline{V_0} \subseteq U_0$ and a strictly increasing sequence $\{n_i\}_{i=1}^\infty \subseteq \mathbb{N}$ such that

$$\phi^{n_i}(x_0) \in V_0 \quad , \quad \forall i \in \mathbb{N}. \tag{2}$$

Choosing, if necessary, a subsequence, we may assume that $n_1 > m_0$ and $n_{i+1} > 3n_i$. By Urysohn's lemma there is $u_0 \in C_0(X)$ such that $u_0(x) = 1$, for all $x \in \overline{V_0}$ and $u_0(X \setminus U_0) = \{0\}$. We thus have

$$u_0(x_0) = u_0 \circ \phi^{n_i}(x_0) = 1 \quad , \quad \forall i \in \mathbb{N}. \tag{3}$$

By [10, Proposition 2.1], we have that $U^{m_0} f_{m_0} \in \mathcal{A}_{hc}$, (see also [7, p. 133]). Hence, if we consider the sequence $\{B_i\}_{i=1}^{\infty}$, where

$$\begin{aligned} B_i &= (U^{n_{i+1}-n_i-m_0} u_0 \circ \phi^{-m_0})(U^{m_0} f_{m_0})(U^{n_i-m_0} u_0 \circ \phi^{-m_0}) \\ &= U^{n_{i+1}-m_0} u_0 \circ \phi^{n_i-m_0} f_{m_0} \circ \phi^{n_i-m_0} u_0 \circ \phi^{-m_0}, \end{aligned}$$

it follows that $\{B_i\}_{i=1}^{\infty} \subseteq \mathcal{A}_{hc}$.

Let $\pi : \mathcal{A}_{hc} \to \mathcal{A}_{hc}/\mathcal{I}$ be the quotient map. To prove that the element $\pi(A)$ is not a compact element of $\mathcal{A}_{hc}/\mathcal{I}$, we will prove that the sequence $\{M_{\pi(A),\pi(A)}(\pi(B_i))\}_{i\in\mathbb{N}}$ has no Cauchy subsequence.

Let $k, l \in \mathbb{N}$ with $k > l$. If $r < (n_{k+1} - m_0)$, the rth Fourier coefficient of B_k is 0, and this also holds for $M_{A,A}(B_k)$. It follows that

$$E_{n_{l+1}+m_0}(M_{A,A}(B_k)) = 0,$$

since $n_{l+1} + m_0 < 3n_{l+1} - m_0 < n_{k+1} - m_0$.

Therefore, it follows that

$$\begin{aligned} \|M_{\pi(A),\pi(A)}(\pi(B_k - B_l))\| &= \inf_{N\in\mathcal{I}} \|M_{A,A}(B_k - B_l) + N\| \\ &\geq \inf_{N\in\mathcal{I}} \|E_{n_{l+1}+m_0}\left(M_{A,A}(B_k - B_l) + N\right)\| \\ &\geq \inf_{N\in\mathcal{I}} |E_{n_{l+1}+m_0}\left(M_{A,A}(B_l) + N\right)(x_0)| \\ &= |E_{n_{l+1}+m_0}(M_{A,A}(B_l))(x_0)| \end{aligned}$$

since $x_0 \in X_r$ and thus, for all $N \in \mathcal{I}$, we have $E_{n_{l+1}+m_0}(N)(x_0) = 0$.

We calculate $|E_{n_{l+1}+m_0}(M_{A,A}(B_l))(x_0)|$.

We have

$$|E_{n_{l+1}+m_0}(M_{A,A}(B_l))(x_0)| = \\ \left|\sum_{n=0}^{2m_0} (f_{2m_0-n} \circ \phi^{n_{l+1}+n-m_0} u_0 \circ \phi^{n_l+n-m_0} f_{m_0} \circ \phi^{n_l+n-m_0} u_0 \circ \phi^{n-m_0} f_n)(x_0)\right|.$$

For $n < m_0$ we have $f_n(x_0) = 0$. Also, for $n > m_0$ and $n \leq 2m_0$ we have $f_{2m_0-n} \circ \phi^{n_{l+1}+n-m_0}(x_0) = 0$, since $2m_0 - n < m_0$ and $\phi^{n_{l+1}+n-m_0}(x_0) \in X_r$.

Finally,

$$|E_{n_{l+1}+m_0}(M_{A,A}(B_l))(x_0)| = \\ |(f_{m_0} \circ \phi^{n_{l+1}} u_0 \circ \phi^{n_l} f_{m_0} \circ \phi^{n_l} u_0 f_{m_0})(x_0)| \geq \frac{|f_{m_0}^3(x_0)|}{8}.$$

It follows that the sequence $\{M_{\pi(A),\pi(A)}(\pi(B_i))\}_{i\in\mathbb{N}}$ contains no Cauchy subsequence, and hence $\pi(A)$ is not a compact element of $\mathcal{A}_{hc}/\mathcal{I}$.

□

3. THE SCATTERED RADICAL

The following are taken from [18, 8.2]. A Banach algebra is called *scattered* if the spectrum of every element $a \in \mathcal{A}$ is finite or countable. A Banach algebra $\mathcal{A}$ has a largest scattered ideal denoted by $\mathcal{R}_s(\mathcal{A})$. This ideal is closed and is called the scattered radical of $\mathcal{A}$ [18, Theorem 8.10].

Since all C^*-algebras are semisimple and their quotients are again C^*-algebras, it follows from [18, Theorem 8.22] that $C_0(X)_{hc} = C_0(X)_s$.

Donsig, Katavolos and Manousos proved in [7] a characterization of the Jacobson radical for more general semicrossed products. The next theorem follows from their result [7, Theorem 18].

Theorem 3.1. *The Jacobson radical of $\mathcal{A}$ coincides with the set of operators*

$$\{A \in \mathcal{A} \mid E_n(A)(X_r) = \{0\},\ \forall n \in \mathbb{Z}_+ \text{ and } E_0(A) = 0\}.$$

It follows from Theorem 2.6 and the above characterization, that the Jacobson radical of $\mathcal{A}$ is contained in $\mathcal{A}_{hc}$. Hence, from [18, Theorem 8.15] we obtain the following.

Theorem 3.2.

$$\mathcal{A}_{\mathrm{hc}} = \mathcal{A}_s.$$

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