ON THE ASYMPTOTIC BEHAVIOR OF CONDENSER CAPACITY UNDER BLASCHKE PRODUCTS AND UNIVERSAL COVERING MAPS

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ABSTRACT. We prove an estimate for the capacity of the condenser $(\mathbb{D}, K_r)$, $r \in (0,1)$, where $\mathbb{D}$ is the open unit disc and $\{K_r\}$ is a compact exhaustion of the inverse image of a compact set under a Blaschke product B, involving weighted logarithmic integral means of the Frostman shifts of B. Also, we describe the asymptotic behavior of the capacity of condensers $(\mathbb{D}, E_r)$, where E_r is a connected component of the inverse image of a closed disc with radius r under universal covering maps, as $r \to 0$.

1. INTRODUCTION

A *condenser* in the complex plane $\mathbb{C}$ is a pair (D,K) where D is a proper subdomain of $\mathbb{C}$ and K is a compact subset of D. The sets K and $\mathbb{C} \setminus D$ are called the plates of the condenser (D,K). Let h be the solution of the generalized Dirichlet problem on $D \setminus K$ with boundary values 0 on ∂D and 1 on ∂K. The function h is the *equilibrium potential* of the condenser (D,K). The *capacity* of (D,K) is

$$\operatorname{Cap}(D,K) = \int_{D\setminus K} |\nabla h(z)|^2 dA(z),$$

where dA is the two-dimensional Lebesgue measure.

In principle, if the two plates of a condenser get closer, its capacity tends to infinity. The rate of growth of the capacity of several types of condensers whose plates are getting closer has been studied in the recent papers [1, 5, 8, 13, 16, 19]. In this paper, we are interested in the rate of growth of the capacity of condensers and how it depends on geometric or analytic quantities expressing the way their plates approach each other. In particular, we will consider condensers $(\mathbb{D}, K_r)$, $r \in (0,1)$, where K_r is an increasing sequence of compact subsets of the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, exhausting the inverse image $f^{-1}(C)$ of a compact set C under certain holomorphic functions f in $\mathbb{D}$ having infinite valency.

A function $\phi : \mathbb{D} \mapsto \mathbb{C}$ is called inner if ϕ is a bounded holomorphic function having unimodular radial boundary limits almost everywhere on the unit circle.

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Every inner function $\phi(z)$, $z \in \mathbb{D}$, can be expressed as a product $\lambda B(z)S(z)$, where $|\lambda| = 1$,

$$B(z) = \prod_{n=1}^{\infty} \frac{\bar{z_n}}{|z_n|} \frac{z_n - z}{1 - \bar{z_n} z}, \qquad z \in \mathbb{D},$$

is the Blaschke product corresponding to the sequence of zeros $\{z_n\} \subset \mathbb{D}$ of ϕ satisfying the Blaschke condition $\sum_{n=1}^{+\infty} 1 - |z_n| < +\infty$ and

$$S(z) = \exp\left(-\int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right), \qquad z \in \mathbb{D},$$

where μ is a (positive) singular measure on $\partial\mathbb{D}$ related to the zeros of ϕ on the unit circle. The following characterization of Blaschke products in the class of all inner functions is well known (see e.g. [15, p. 18]). An inner function ϕ is a Blaschke product if and only if

$$\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \log|\phi(re^{i\theta})| d\theta = 0.$$

The rate of growth of the above integral means of $\log|\phi(re^{i\theta})|$ to 0 as $r \to 1$ gives rise to different families of Blaschke products. Let B be a Blaschke product and let

$$T(r) = T(r,B) := \frac{1}{2\pi \log r} \int_0^{2\pi} \log|B(re^{i\theta})| d\theta, \qquad r \in (0,1).$$

B is called an exponential Blaschke product if there exists a positive integer $M = M(B)$ such that, for every $n \in \mathbb{N}$, B has at most M zeros on the annulus $\{z \in \mathbb{D} : 2^{-n-1} \leq 1 - |z| \leq 2^{-n}\}$ counting multiplicities. Then (see [6, Theorem 2]) B is an exponential Blaschke product if and only if

$$\sup_{n \in \mathbb{N}} |T(1 - 2^{-n-1}) - T(2^{-n})| < +\infty.$$

Also, $\sup\{T(r) : r \in (0,1)\} < +\infty$ if and only if B has finitely many zeros. For more results concerning the relation between the integral means

$$\left(\frac{1}{2\pi} \int_0^{2\pi} \left| \log|B(re^{i\theta})| \right|^p d\theta \right)^{1/p}, \qquad p \geq 1, r \in (0,1),$$

and the distribution of zeros of B see [10] and references therein.

We will examine the asymptotic behavior of the capacity of inverse images of condensers under Blaschke products. Let $D(a,s) = \{z \in \mathbb{C} : |z - a| < s\}$ be the open disc centered at $a \in \mathbb{C}$ and having radius $s > 0$. The following result has been proved in [16].

Theorem A. [16, p. 3552] *Let B be an exponential Blaschke product, let $(\mathbb{D}, C)$ be a condenser with positive capacity and let $K_n = B^{-1}(C) \cap \overline{D(0, 1 - 2^{-n})}$. Then*

$$\mathrm{Cap}(\mathbb{D}, K_n) = \mathcal{O}(n), \qquad \text{as } n \to +\infty.$$

In our first main result we will give an estimate for the rate of growth of the capacity of the inverse image of a condenser $(\mathbb{D}, C)$ under an arbitrary Blaschke product, in terms of the integral means of its Frostman shifts. Let B be a Blaschke product and let

$$B_y(z) = \frac{B(z) - y}{1 - \overline{y}B(z)}, \qquad z \in \mathbb{N}, y \in \mathbb{D},$$

be the family of Frostman shifts of B. By Frostman's Theorem (see e.g. [15, p. 35]), B_y is a Blaschke product for every $y \in \mathbb{D}$ except on a set of zero logarithmic capacity. In particular, the set $\mathbb{D} \setminus B(\mathbb{D})$ has zero logarithmic capacity. For every $y \in \mathbb{D}$, let $T(r,y,B) := T(r,B_y)$ and let

$$T(r,C,B) := \sup_{y \in C} T(r,y,B).$$

In the following theorem we give an estimate for the rate of growth of the capacity of the inverse image of a condenser under an arbitrary Blaschke product.

Theorem 1.1. *Let B be a Blaschke product, let $(\mathbb{D},C)$ be a condenser with positive capacity and let*

$$K_r = B^{-1}(C) \cap \overline{D(0,r)}, \qquad r \in (0,1).$$

Then

$$\mathrm{Cap}(\mathbb{D},K_r) = \mathscr{O}(T(r(1-\log r),C,B)), \qquad \text{as } r \to 1.$$

Using Theorem 1.1 we obtain the following corollary about the asymptotic behavior of the capacity of the condensers $(\mathbb{D},K_r)$ for Blaschke products with Frostman shifts having zero sequences of restricted growth. For every $y \in \mathbb{D}$ and $r \in (0,1)$, let $N(r,y,B)$ be the number of zeros of the Frostman shift B_y of B on the closed disc $\overline{D(0,r)}$, counting multiplicities.

Corollary 1.2. *Let B be a Blaschke product, let $(\mathbb{D},C)$ be a condenser with positive capacity and let*

$$K_r = B^{-1}(C) \cap \overline{D(0,r)}, \qquad r \in (0,1).$$

Let $s \in (0,1)$ and suppose that

$$N(r,C,B) := \sup_{y \in C} N(r,y,B) = \mathscr{O}((1-r)^{-s}). \tag{1.1}$$

Then

$$\mathrm{Cap}(\mathbb{D},K_r) = \mathscr{O}((1-r+r\log r)^{-s}), \qquad \text{as } r \to 1.$$

Another type of holomorphic functions on the unit disk, with infinite valency, that we will consider when taking inverse images of compact sets are the universal covering maps of multiply connected domains. A holomorphic function $f : \mathbb{D} \mapsto D$ is called a universal covering map if for every $z \in D$, there exists $r > 0$ such that

$$f^{-1}(D(z,r)) = \bigcup_i A_i,$$

where A_i are non empty disjoint subdomains of $\mathbb{D}$ and f maps homeomorphically A_i onto $D(z,r)$, for every i; the disc $D(z,r)$ is called a fundamental neighborhood of $z \in D$. For more information about the properties of universal covering maps see e.g. [7].

Let $D \subset \mathbb{C}$ be a Greenian domain; that is, D has a Green function or, equivalently, the complement of D has positive logarithmic capacity (see section 2.4). In particular, every Greenian domain is hyperbolic; that is, its boundary contains at least two points. Let $f : \mathbb{D} \mapsto D$ be a universal covering map of D and let $D(z,s)$ be a fundamental neighborhood of $z \in D$. Let $r \in (0,s)$, let $E_i(z,r)$, $i \in \mathbb{N}$, be an enumeration of the connected components of $f^{-1}(\overline{D(z,r)})$ and let $K_n(z,r) = \cup_{i=1}^{n} E_i(z,r)$, $n \in \mathbb{N}$. Note that, from the conformal equivalence of $E_i(z,r)$ and the conformal invariance of condenser capacity,

$$\text{Cap}(\mathbb{D}, E_i(z,r)) = \text{Cap}(\mathbb{D}, E_1(z,r)), \qquad i \in \mathbb{N}. \tag{1.2}$$

It has been proved in [16, p. 3554] that for every $n \in \mathbb{N}$,

$$\text{Cap}(D, \overline{D(z,r)}) \leq \frac{\text{Cap}(\mathbb{D}, K_n(z,r))}{n} \leq \text{Cap}(\mathbb{D}, E_1(z,r)). \tag{1.3}$$

Also, if the domain D is doubly connected (see [16, p. 3556]),

$$\text{Cap}(\mathbb{D}, \overline{D(z,r)}) = \lim_{n\to\infty} \frac{\text{Cap}(\mathbb{D}, K_n(z,r))}{n}. \tag{1.4}$$

The equality (1.4) shows that the first inequality in (1.3) is sharp. From the following theorem it follows that the second inequality in (1.3) is asymptotically sharp as $r \to 0$.

Theorem 1.3. *Let $D \subset \mathbb{C}$ be a Greenian domain, let $f : \mathbb{D} \mapsto D$ be a universal covering map and let $z \in D$. For $0 < r < \text{dist}(z, \partial D)$, let $E(z,r)$ be a connected component of $f^{-1}(\overline{D(z,r)})$. Then,*

$$\lim_{r\to 0} \frac{\text{Cap}(\mathbb{D}, E(z,r))}{\text{Cap}(D, \overline{D(z,r)})} = 1. \tag{1.5}$$

Note that, from (1.2), the equality (1.5) does not depend on the choice of the connected component $E(z,r)$.

In the following section we collected several known definitions and statements that will be used in the proofs of our main results. In section 3 we prove Theorem 1.1 and show how it implies Theorem A and Corollary 1.2. Theorem 1.3 is proved in section 4.

2. Background Material

2.1. **Green energy and equilibrium measure.** Let (D,K) be a condenser. If D is a Greenian domain, the *Green equilibrium energy* of (D,K) is defined by

$$I(D,K) = \inf_{\mu} \iint G_D(z,w)\, d\mu(z)\, d\mu(w),$$

where $G_D(x,y)$ is the Green function of D and the infimum is taken over all probability Borel measures μ supported on K. When $I(D,K) < +\infty$, the unique probability Borel measure μ_K for which the above infimum is attained is the *Green equilibrium measure*. The function

$$U^D_{\mu_K}(z) = \int G_D(z,w)\, d\mu_K(w), \qquad z \in D,$$

is the *Green equilibrium potential* of (D,K). It is true that $U^D_{\mu_K} = I(D,K)$ on K except on a set of zero logarithmic capacity and

$$U^D_{\mu_K} \leq I(D,K) \tag{2.1}$$

on D; see e.g. [14, p. 174]. From the formula (see [14, p. 97])

$$\iint G_D(z,w)d\mu(z)d\mu(w) = \frac{1}{2\pi}\int_D |\nabla U^D_\mu(z)|^2 dm_2(z),$$

we obtain that

$$\mathrm{Cap}(D,K) = \frac{2\pi}{I(D,K)}. \tag{2.2}$$

For more information about potential theory, see e.g. [2, 14, 18].

2.2. **Lindelöf Principle.** Let f be a non-constant holomorphic function on a Greenian domain D such that $f(D)$ is Greenian. We denote by $m(a)$ the *multiplicity* of the zero of $f(z)-f(a)$ at $a\in D$ and by

$$\mathrm{v}(y) = \sum_{f(a)=y} m(a)$$

the *valency* of f at $y\in f(D)$. The following inequality is known as the Lindelöf Principle (see e.g. [12])

$$G_{f(D)}(y_0, f(z)) \geq \sum_{f(a)=y_0} m(a) G_D(a,z), \tag{2.3}$$

where $z\in D$ and $y_0\in f(D)$. For fixed $y_0\in f(D)$, if equality holds in (2.3) for a point $z\in D$ with $f(z)\neq y_0$, then it holds for every point in D. From the definition of Blaschke products and Frostman's Theorem it follows that equality holds in the Lindelöf Principle for inner functions $\phi:\mathbb{D}\mapsto\mathbb{D}$, for every $y_0\in\mathbb{D}$ except on a set of zero logarithmic capacity. For a characterization of the equality cases in Lindelöf's Principle see [4, Theorem 3].

2.3. **A condenser capacity inequality.** Let (D,K) be a condenser, let f be a non-constant holomorphic function on the domain D such that the condenser $(f(D), f(K))$ has positive capacity, let ν be the Green equilibrium measure of $(f(D), f(K))$ and let $E := \mathrm{supp}(\nu)\setminus f(\{a\in K : m(a)\geq 2\})$. For every $y\in E$, let $N_f(y,K)$ be the cardinality of the set $\{x\in K : f(x)=y\}$. If $V_f(K) := \min_{y\in E} N_f(y,K)$, then

$$\mathrm{Cap}(f(D), f(K)) \leq \frac{\mathrm{Cap}(D,K)}{V_f(K)}, \tag{2.4}$$

see [17, Theorem 3.1].

2.4. **Condenser capacity and logarithmic integrals.** Let K be a compact subset of $\mathbb{C}$ and let $P(K)$ denote the family of Borel probability measures on K. The logarithmic capacity of K is defined by

$$c(K) := \exp\Big(-\inf_{\mu\in P(K)} \iint \log\frac{1}{|z-w|} d\mu(z)d\mu(w)\Big).$$

When $c(K)>0$, there exists a unique measure $\mu_K \in P(K)$, called the equilibrium measure of K, for which the infimum in the above formula is attained. Logarithmic capacity is a well studied notion (see e.g. [18]) which can be expressed via condenser capacity by (see e.g. [9, p. 23])

$$c(K) = \exp\Big(-\lim_{r\to+\infty}\Big(\frac{2\pi}{\operatorname{Cap}(D(0,r),K)} - \log r\Big)\Big), \tag{2.5}$$

for all compact sets $K \subset \mathbb{C}$.

Let (D,K) be a condenser in $\mathbb{C}$ and let $S(D,K)$ denote the family of signed measures $\sigma = \sigma_D - \sigma_K$, where $\sigma_D \in P(\mathbb{C}\setminus D)$ and $\sigma_K \in P(K)$. An alternative formula via logarithmic energy integrals for the Green equilibrium energy of (D,K), due to Bagby [3], is given by

$$I(D,K) = \inf_{\sigma\in S(D,K)} \iint \log\frac{1}{|z-w|} d\sigma(z)d\sigma(w). \tag{2.6}$$

Remark 2.1. *From (2.6) it follows that* $\operatorname{Cap}(D,K)>0$ *if and only if* D *is a Greenian domain and* $c(K)>0$.

2.5. **Hyperbolic metric.** Let $D\subset\mathbb{C}$ be a domain whose boundary contains at least two points, let $z\in D$ and let $f:\mathbb{D}\mapsto D$ be a universal covering map with $f(0)=z$. The density of the hyperbolic metric of D at $z\in D$ is defined by

$$\lambda_D(z) = \frac{1}{|f'(0)|}.$$

From the principle of the hyperbolic metric (see e.g. [11, p. 682]) it follows that

$$\lambda_D(z) \le \lambda_{D(z,\operatorname{dist}(z,\partial D))}(z) = \frac{1}{\operatorname{dist}(z,\partial D)}. \tag{2.7}$$

3. Blaschke products and condenser capacity

In this section we will prove Theorem 1.1, Corollary 1.2 and show how Theorem A follows from Theorem 1.1.

Note that, from the inequality (2.4), $\operatorname{Cap}(\mathbb{D},K_r)\to+\infty$ as $r\to1$, for every Blaschke product having infinitely many zeros.

Proof of Theorem 1.1. For every $y\in\mathbb{D}$ and $r\in(0,1)$, let $T(r,y)=T(r,y,B)$, $N(r,y)=N(r,y,B)$, $E(r,y)=B^{-1}(y)\cap\overline{D(0,r)}$ and $F(r,y)=B^{-1}(y)\setminus E(r,y)$. It is well known (see e.g. [6]) that for every $r\in(0,1)$,

$$T(r,0) = N(r,0) - \frac{1}{\log r}\sum_{a\in F(r,0)} G_{\mathbb{D}}(a,0).$$

Applying the above equality for B_y, for all $y \in \mathbb{D}$ such that B_y is a Blaschke product, we get

$$T(r,y) = N(r,y) - \frac{1}{\log r} \sum_{a \in F(r,y)} G_{\mathbb{D}}(a,0), \tag{3.1}$$

from which it follows that

$$N(r,y) \leq T(r,y) \tag{3.2}$$

and

$$\sum_{a \in F(r,y)} G_{\mathbb{D}}(a,0) \leq (-\log r) T(r,y), \tag{3.3}$$

for every $r \in (0,1)$. If $y \in \mathbb{D}$ is a point for which B_y has no zeros, inequalities (3.2) and (3.3) hold trivially. If $y \in \mathbb{D}$ is a point for which B_y is not a Blaschke product and has a Blaschke factor $\tilde{B_y}$, then the inequalities (3.2) and (3.3) follow from the corresponding inequalities when applied for $\tilde{B_y}$ and from the inequality $|B_y(z)| \leq |\tilde{B_y}(z)|$. So, the inequalities (3.2) and (3.3) hold for all $y \in \mathbb{D}$ and for every $r \in (0,1)$.

For every $r \in (0,1)$, let μ_r be the equilibrium measure of the condenser $(\mathbb{D}, K_r)$ and consider the measure

$$\nu_r(A) = \mu_r(B^{-1}(A)), \qquad A \subset C \text{ Borel measurable.}$$

Then ν_r is a probability Borel measure on C, for every $r \in (0,1)$. Therefore,

$$I(\mathbb{D}, C) \leq \iint G_{\mathbb{D}}(x,y) d\nu_r(x) d\nu_r(y), \qquad \text{for every } r \in (0,1). \tag{3.4}$$

Note that, since $\mathbb{D} \setminus B(\mathbb{D})$ has zero logarithmic capacity, $G_{B(\mathbb{D})}(x,y) = G_{\mathbb{D}}(x,y)$, for all $x, y \in B(\mathbb{D})$. From the Lindelöf Principle,

$$\begin{aligned}
\iint G_{\mathbb{D}}(x,y) d\nu_r(x) d\nu_r(y) &= \iint G_{\mathbb{D}}(B(z),y) d\mu_r(z) d\nu_r(y) \\
&= \iint \sum_{a \in E(r - r\log r, y)} G_{\mathbb{D}}(z,a) d\mu_r(z) d\nu_r(y) \\
&\quad + \iint \sum_{a \in F(r - r\log r, y)} G_{\mathbb{D}}(z,a) d\mu_r(z) d\nu_r(y).
\end{aligned}$$

We will estimate the two integrals in the above sum separately. From the inequalities (2.1) and (3.2),

$$\begin{aligned}
\iint \sum_{a\in E(r-r\log r,y)} G_{\mathbb{D}}(z,a)d\mu_r(z)d\nu_r(y) &= \int \sum_{a\in E(r-r\log r,y)} \int G_{\mathbb{D}}(z,a)d\mu_r(z)d\nu_r(y)\\
&= \int \sum_{a\in E(r-r\log r,y)} U^{\mathbb{D}}_{\mu_r}(a)d\nu_r(y)\\
&\le \int \sum_{a\in E(r-r\log r,y)} I(\mathbb{D},K_r)d\nu_r(y)\\
&= \int N(r-r\log r,y)d\nu_r(y)I(\mathbb{D},K_r)\\
&\le \int T(r-r\log r,y)d\nu_r(y)I(\mathbb{D},K_r)\\
&\le T(r-r\log r,C,B)I(\mathbb{D},K_r), \qquad (3.5)
\end{aligned}$$

since ν_r is a probability measure.

We will now estimate the second integral. For all $a\in E(r-r\log r,y)$ and $y\in C$, the function $z\mapsto G_{\mathbb{D}}(z,a)$ is a positive harmonic function on the disc $D(0,r-r\log r)$. Applying Harnack's inequality (see [18, p. 14]), we get that for every $z\in K_r$, $a\in E(r-r\log r,y)$ and $y\in C$,

$$\begin{aligned}
G_{\mathbb{D}}(z,a) &\le \frac{r-r\log r+|z|}{r-r\log r-|z|}G_{\mathbb{D}}(0,a)\\
&\le \frac{2r-r\log r}{-r\log r}G_{\mathbb{D}}(0,a). \qquad (3.6)
\end{aligned}$$

From the monotonicity property of condenser capacity,

$$I(\mathbb{D},K_r)\ge I(\mathbb{D},\overline{D(0,r)})=-\log r. \qquad (3.7)$$

Since μ_r is a probability measure, from the inequalities (3.6), (3.3) and (3.7) we get

$$\begin{aligned}
&\iint \sum_{a\in F(r-r\log r,y)} G_{\mathbb{D}}(z,a)d\mu_r(z)d\nu_r(y)\\
\le\ & \frac{2r-r\log r}{-r\log r}\int \sum_{a\in F(r-r\log r,y)} G_{\mathbb{D}}(0,a)d\nu_r(y)\\
\le\ & \frac{-(2r-r\log r)\log(r-r\log r)}{-r\log r}\int T(r-r\log r,y)d\nu_r(y)\\
\le\ & -2\log r\int T(r-r\log r,C,B)d\nu_r(y)\\
\le\ & 2I(\mathbb{D},K_r)T(r-r\log r,C,B). \qquad (3.8)
\end{aligned}$$

Finally, from the inequalities (3.4), (3.5) and (3.8) we obtain that

$$\mathrm{Cap}(\mathbb{D},K_r)\le \frac{3}{I(D,C)}T(r-r\log r,C,B),$$

from which the conclusion follows. □

Proof of Corollary 1.2. Eiko and Kondratyuk [10, p. 170] showed that the condition (1.1) implies that there exist $L > 0$ and $r_0 \in (0,1)$ such that

$$\frac{-1}{2\pi}\int_0^{2\pi} \log|B_y(re^{i\theta})|d\theta \leq L(1-r)^{1-s},$$

for every $y \in C$ and $r \in (r_0,1)$. Therefore

$$T(r,C,B) = \mathscr{O}((1-r)^{-s}),$$

as $r \to 1$ and the conclusion follows from Theorem 1.1. □

Proof of Theorem A. Since B is an exponential Blaschke product, it follows from [6, p. 587] and [16, Lemma 3.2, p. 3551] that

$$R := \sup\Big\{-\frac{1}{\log r}\sum_{a\in F(r,y)} G_{\mathbb{D}}(a,0) : y \in C \text{ and } r \in (1/2,1)\Big\} < +\infty.$$

Therefore, from the equality (3.1), we obtain that

$$T(r,y) \leq N(r,y) + R \leq (1+R)N(r,y),$$

for every $y \in C$ and for all r sufficiently close to 1. Therefore

$$T(r,C,B) = \mathscr{O}(N(r,C,B)), \qquad \text{as } r \to 1.$$

From the definition of exponential Blaschke products and [16, Lemma 3.2, p. 3551] we obtain that there exist $M > 0$ such that $N(1-2^{-n},C,B) \leq nM$, for every $n \in \mathbb{N}$. Also, since $(1-2^{-n})(1-\log(1-2^{-n})) \leq 1-2^{-3n}$ for every $n \in \mathbb{N}$, we get that

$$N((1-2^{-n})(1-\log(1-2^{-n})),C,B) = \mathscr{O}(n), \qquad \text{as } n \to +\infty,$$

and the conclusion follows from Theorem 1.1. □

4. Universal covering maps and condenser capacity

In this section we will prove Theorem 1.3.

Proof of Theorem 1.3. Using the conformal invariance of condenser capacity and an auxiliary automorphism of $\mathbb{D}$, we can assume that $f(0) = z$. From the inequality (2.7)

$$|f'(0)| = \frac{1}{\lambda_D(z)} \geq \operatorname{dist}(z,\partial D).$$

Therefore, there exists $\delta > 0$ such that

$$\Big|\frac{f(0)-f(w)}{w}\Big| \geq \frac{\operatorname{dist}(z,\partial D)}{2},$$

for every $w \in D(0,\delta)\setminus\{0\}$. For $0 < r < \operatorname{dist}(z,\partial D)$, choose $E(z,r)$ to be the connected component of $f^{-1}(\overline{D(z,r)})$ containing 0 and let $r_0 > 0$ be such that $E(z,r) \subset D(0,\delta)$, for $r \in (0,r_0)$. Then, for all $w \in \partial E(z,r)$ and $r \in (0,r_0)$,

$$\frac{r}{|w|} = \Big|\frac{f(0)-f(w)}{w}\Big| \geq \frac{\operatorname{dist}(z,\partial D)}{2}.$$

It follows that

$$E(z,r) \subset D\Big(0, \frac{2r}{\text{dist}(z,\partial D)}\Big)$$

and

$$\text{Cap}(\mathbb{D}, E(z,r)) \le \text{Cap}\Big(\mathbb{D}, \overline{D\Big(0, \frac{2r}{\text{dist}(z,\partial D)}\Big)}\Big) = \frac{2\pi}{\log \frac{\text{dist}(z,\partial D)}{2r}}, \tag{4.1}$$

for all $r \in (0, r_0)$.

In order to get a lower bound for the capacity of the condenser $(D, \overline{D(z,r)})$ we will use the formula (2.6). Let E be a compact subset of $\mathbb{C} \setminus D$ with positive logarithmic capacity and let τ_E be the equilibrium measure of E. Also, for $0 < r < \text{dist}(z,\partial D)$, let τ_r be the equilibrium measure of the disc $\overline{D(z,r)}$; that is, τ_r is the normalized Lebesgue measure on $\partial D(z,r)$. We will estimate the Green equilibrium energy of $(D, \overline{D(z,r)})$ using the measure $\tau = \tau_E - \tau_r \in S(D, \overline{D(z,r)})$. If d is the diameter of the compact set $E \cup \overline{D(z,r)}$, then

$$\begin{aligned}
I(D, \overline{D(z,r)}) &\le \iint \log \frac{1}{|z-w|} d\tau(z) d\tau(w) \\
&= \iint \log \frac{1}{|z-w|} d\tau_E(z) d\tau_E(w) + \iint \log \frac{1}{|z-w|} d\tau_r(z) d\tau_r(w) \\
&\quad -2 \iint \log \frac{1}{|z-w|} d\tau_E(z) d\tau_r(w) \\
&\le \log \frac{1}{c(E)} + \log \frac{1}{r} - 2 \log \frac{1}{d} \\
&= \log \frac{d^2}{rc(E)}.
\end{aligned}$$

Therefore,

$$\text{Cap}(D, \overline{D(z,r)}) \ge \frac{2\pi}{\log \frac{d^2}{rc(E)}}. \tag{4.2}$$

From the inequalities (4.1) and (4.2) we get

$$1 \le \liminf_{r \to 0} \frac{\text{Cap}(\mathbb{D}, E(z,r))}{\text{Cap}(D, \overline{D(z,r)})} \le \limsup_{r \to 0} \frac{\text{Cap}(\mathbb{D}, E(z,r))}{\text{Cap}(D, \overline{D(z,r)})} \le \lim_{r \to 0} \frac{\log \frac{d^2}{rc(E)}}{\log \frac{\text{dist}(z,\partial D)}{2r}} = 1,$$

and (1.5) follows.

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