SEMIGROUPS OF HOLOMORPHIC FUNCTIONS AND CONDENSER CAPACITY

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Abstract. Suppose $(\phi_t)_{t\geq 0}$ is a semigroup of holomorphic functions in the unit disk $\mathbb{D}$ with Denjoy-Wolff point $\tau = 1$. Suppose K is a compact subset of $\mathbb{D}$. We prove that the capacity of the condenser $(\mathbb{D}, \phi_t(K))$ is a decreasing function of t. Moreover, we study its asymptotic behavior as $t \to +\infty$ in relation with the type of the semigroup.

1 Introduction

A one-parameter family $(\phi_t)_{t\geq 0}$ of holomorphic functions in the unit disk is a semigroup if the following conditions hold.

(i) $\phi_0(z) = z$, for all $z \in \mathbb{D}$

(ii) $\phi_{t+s}(z) = \phi_t(\phi_s(z))$, for every $t, s \geq 0$ and $z \in \mathbb{D}$

(iii) $\lim_{t\to s} \phi_t(z) = \phi_s(z)$, for all $s \geq 0$ and $z \in \mathbb{D}$.

It also follows that for every $t > 0$, the function ϕ_t is univalent.

The introduction of semigroups of holomorphic functions was made by Berkson and Porta in [4]. For the theory and applications of the semigroups, the reader may refer to [1], [5], [7], [9] and [14]. A basic property of (ϕ_t) is that there exists a unique point $\tau \in \overline{\mathbb{D}}$ (the Denjoy-Wolff point of the semigroup) such that for every $z \in \mathbb{D}$,

$$\lim_{t\to+\infty} \phi_t(z) = \tau; \tag{1.1}$$

see [1, Theorem 1.4.17]. The point τ coincides with the Denjoy-Wolff point of ϕ_1. If $\tau \in \mathbb{D}$ and ϕ_t is not an elliptic automorphism of $\mathbb{D}$ for any $t \geq 0$, then (ϕ_t) is an *elliptic semigroup* of holomorphic functions.

In the present work, we are interested in the case where the Denjoy-Wolff point is a boundary point of the unit disk. Without loss of generality, we can and do assume that $\tau = 1$.

The Denjoy-Wolff point of the semigroup is a fixed point for every function ϕ_t, $t \geq 0$; namely

$$\angle \lim_{z\to 1} \phi_t(z) = 1. \quad \text{(angular limit)}$$

The angular derivative of ϕ_t at 1 is

$$\phi_t'(1) := \angle \lim_{z\to 1} \frac{\phi_t(z) - 1}{z - 1} \leq 1;$$

see [1], [13]. If $\phi_t'(1) < 1$, then (ϕ_t) is called a *hyperbolic semigroup*. If $\phi_t'(1) = 1$, for some (hence for all) t, then (ϕ_t) is a *parabolic semigroup*. For semigroups (ϕ_t) with Denjoy-Wolff point 1, there exists a conformal mapping $h : \mathbb{D} \to \mathbb{C}$ with $h(0) = 0$, which satisfies

$$\phi_t(z) = h^{-1}(h(z) + t). \tag{1.2}$$

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for every $z \in \mathbb{D}$; see [1, Theorem 1.4.22]. This mapping h is called the *Koenigs function* of the semigroup.

Moreover, the simply connected domain $\Omega := h(\mathbb{D})$ has the following geometric property. If $w \in \Omega$, then $\{w + s : s > 0\} \subset \Omega$ by (1.2). By [7, Theorem 2.1], the semigroup is hyperbolic if and only if Ω is contained in a horizontal strip. Otherwise, (ϕ_t) is parabolic.

Fix $z \in \mathbb{D}$. The curve $\gamma_z : [0, +\infty) \to \mathbb{D}$ with $\gamma_z(t) = \phi_t(z)$ is called *trajectory* of z. The image of the trajectory under the Koenigs function of the semigroup is the curve (half-line)

$$h(\gamma_z(t)) = h(z) + t,$$

for $t \geq 0$. Note that as $t \to +\infty$, $\gamma_z(t)$ tends asymptotically to 1, due to (1.1).

There exists a further partition of the class of parabolic semigroups. A parabolic semigroup is of *zero hyperbolic step*, if for any $s > 0$ and any $z \in \mathbb{D}$

$$\lim_{t \to +\infty} d_{\mathbb{D}}(\phi_t(z), \phi_{t+s}(z)) = 0,$$

where $d_{\mathbb{D}}$ denotes the hyperbolic distance; see section 2.3. If the limit is not equal to zero for some $s > 0$ or $z \in \mathbb{D}$, then the parabolic semigroup is of *positive hyperbolic step*. In that case, the domain Ω is contained in a horizontal half-plane, as stated in [5, Theorem 1]. This does not occur when the semigroup is of zero hyperbolic step.

Suppose K is a compact subset of $\mathbb{D}$ with positive logarithmic capacity. The *trajectory* of K is the family of compact sets

$$\gamma_K(t) := \bigcup_{z \in K} \gamma_z(t) = \bigcup_{0 \leq s \leq t} \phi_s(K), \quad 0 \leq t < +\infty,$$

which consist of all trajectories of $z \in K$.

As we have stated, due to (1.1), $\gamma_t(z)$ tends asymptotically to the Denjoy-Wolff point 1 of the semigroup. A question that arises is what happens to the trajectory of K, as $t \to +\infty$. Intuitively, we can say that $\phi_t(K)$ is getting 'smaller' and approaches the Denjoy-Wolff point.

We notice, at this point, that for a fixed t, the pair $(\mathbb{D}, \phi_t(K))$ forms a condenser, as $\mathbb{D}$ is a proper domain of $\mathbb{C}$ and $\phi_t(K) \subset \mathbb{D}$ is compact; see Section 2.2. A natural way to measure the size of a condenser is its capacity.

Therefore, a natural way to study the above question is to examine the capacity of the condenser $(\mathbb{D}, \phi_t(K))$. In this direction, we obtain the following result.

Theorem 1.1. *Let K be a compact subset of $\mathbb{D}$ and $(\phi_t)_{t \geq 0}$ be a semigroup of holomorphic functions in $\mathbb{D}$. The capacity of the condenser $(\mathbb{D}, \phi_t(K))$ is a strictly decreasing function of $t \geq 0$, unless ϕ_{t_0} is an automorphism of $\mathbb{D}$ for some $t_0 > 0$. In this case, the capacity of the condenser $(\mathbb{D}, \phi_t(K))$ is constant for every $t \geq 0$.*

Since $\operatorname{cap}(\mathbb{D}, \phi_t(K))$ is decreasing, its limit as $t \to +\infty$ exists. We examine the asymptotic behavior of the capacity of the condenser $(\mathbb{D}, \phi_t(K))$ in relation to the type of the semigroup.

Theorem 1.2. *Let K be a compact subset of $\mathbb{D}$ and $(\phi_t)_{t \geq 0}$ be a hyperbolic semigroup. Then*

$$\lim_{t \to +\infty} \operatorname{cap}(\mathbb{D}, \phi_t(K)) = \operatorname{cap}(S, h(K)),$$

where h is the associated Koenigs function of the semigroup and S is the smallest horizontal strip that contains $h(\mathbb{D})$.

Theorem 1.3. *Let K be a compact subset of $\mathbb{D}$ and $(\phi_t)_{t \geq 0}$ be a parabolic semigroup. If (ϕ_t) is of zero hyperbolic step, then*

$$\lim_{t \to +\infty} \operatorname{cap}(\mathbb{D}, \phi_t(K)) = 0.$$

If (ϕ_t) is of positive hyperbolic step, then

$$\lim_{t\to+\infty} \operatorname{cap}(\mathbb{D}, \phi_t(K)) = \operatorname{cap}(H, h(K)),$$

where h is the associated Koenigs function of the semigroup and H is the smallest horizontal half-plane that contains $h(\mathbb{D})$.

Hence, we see that the capacity of the condenser $(\mathbb{D}, \phi_t(K))$ has a direct connection with the domain Ω, and therefore with the type of the semigroup.

2 Preparation for the proofs

2.1 **Logarithmic capacity** Let K be a compact subset of $\mathbb{C}$. The euclidean n-th diameter of K is

$$d_n(K) = \sup_{w_\mu, w_\nu \in K} \prod_{1\le\mu<\nu\le n} |w_\mu - w_\nu|^{\frac{2}{n(n-1)}} \tag{2.1}$$

and the supremum is attained, since K is compact, for a n-tuple of points, which is called *Fekete n-tuple*; see [12, Definition 5.5.1].

The *logarithmic capacity* of K is the limit

$$\operatorname{cap} K = \lim_{n\to+\infty} d_n(K).$$

Sets of zero logarithmic capacity are called *polar sets* and they are negligible in the view of potential theory.

2.2 **Capacity of condensers** A *condenser* is an ordered pair (G, E), where G is a proper domain of $\widehat{\mathbb{C}}$ and E is a compact subset of G. If both ∂G and E have strictly positive logarithmic capacity, then the *capacity* of a condenser is defined as

$$\operatorname{cap}(G, E) = \int_{G\setminus E} |\nabla v(z)|^2 dA(z),$$

where A is the Lebesgue measure on the complex plane. The function v is the solution of the generalized Dirichlet problem on $G \setminus E$, with boundary values 0 on ∂G and 1 on ∂E and it is called *equilibrium potential* of the condenser.

We will need some properties for condenser capacity. First of all, we say that the condenser (G_1, E_1) is contained in the condenser (G_2, E_2) if $G_1 \subset G_2$ and $E_2 \subset E_1$, and we write $(G_1, E_1) \subset (G_2, E_2)$. Then their capacities have the following relation.

Lemma 2.1. [8, Theorem 1.8] *If $(G_1, E_1) \subset (G_2, E_2)$, then*

$$\operatorname{cap}(G_1, E_1) \ge \operatorname{cap}(G_2, E_2).$$

If $\{(G_n, E_n)\}_n$ is an increasing sequence of condensers, with $\cap_{n\in\mathbb{N}} E_n := E$ and $\cup_{n\in\mathbb{N}} G_n := G$, we say that it forms an *exhaustion* of the condenser (G, E).

Lemma 2.2. [8, Theorem 1.11] *If $\{(G_n, E_n)\}_n$ is an exhaustion of the condenser (G, E), then*

$$\lim_{n\to+\infty} \operatorname{cap}(G_n, E_n) = \operatorname{cap}(G, E).$$

An important property of condenser capacity is conformal invariance. If f is a conformal map on G, then

$$\operatorname{cap}(G, E) = \operatorname{cap}(f(G), f(E)). \tag{2.2}$$

Suppose $\alpha \in \mathbb{C}$ and $0 < r < s$. We consider the disks $D(\alpha, r)$ and $D(\alpha, s)$ centered at α of radius r and s, respectively.

The capacity of the condenser $(D(\alpha,s),\overline{D(\alpha,r)})$ is

$$\operatorname{cap}(D(\alpha,s),\overline{D(\alpha,r)}) = 2\pi\left(\log\frac{s}{r}\right)^{-1}. \tag{2.3}$$

More information on condenser capacity can be found in [8].

2.3 **Hyperbolic metric** The hyperbolic metric in $\mathbb{D}$ is

$$\lambda_{\mathbb{D}}(z)|dz| := \frac{|dz|}{1-|z|^2},$$

where $\lambda_{\mathbb{D}}$ denotes its density. Let U be a simply connected domain of $\mathbb{C}$. The density of the hyperbolic metric on U is equal to

$$\lambda_U(z) = \lambda_{\mathbb{D}}(f(z))\left|f'(z)\right|,$$

where f is a conformal mapping of U onto $\mathbb{D}$. The hyperbolic metric on U is independent of the choice of the conformal map.

The hyperbolic distance between two points $a,\, b \in U$ is

$$d_U(a,b) = \inf_{\gamma\subset U}\int_\gamma \lambda_U(z)|dz|,$$

where γ is any rectifiable curve that lies in U and joins a,b. The infimum is attained for the hyperbolic geodesic arc that joins a,b.

The hyperbolic geodesic curves of the unit disk $\mathbb{D}$ are the arcs of euclidean circles in $\mathbb{D}$ that are orthogonal to the boundary. Moreover, the hyperbolic distance in the unit disk, for $a,b\in\mathbb{D}$, is defined by

$$d_{\mathbb{D}}(a,b) = \operatorname{arctanh}\left|\frac{a-b}{1-\bar{a}b}\right|.$$

The hyperbolic distance is invariant under any conformal automorphism of $\mathbb{D}$. It is known that every automorphism T of the disk can be represented by

$$T(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z},$$

for $\alpha\in\mathbb{D}$ and $\theta\in\mathbb{R}$. We denote the set of all conformal automorphisms of $\mathbb{D}$ by $\operatorname{Aut}(\mathbb{D})$.

Furthermore, the hyperbolic distance is invariant under conformal mappings. More specifically, if $f: U\to\mathbb{D}$ is conformal, then

$$d_U(z,w) = d_{\mathbb{D}}(f(z),f(w)),$$

for every choice of $z,w\in U$. We refer to [3] for further properties of the hyperbolic metric.

2.4 **Green function-Green capacity** Let D be a domain of the extended complex plane $\widehat{\mathbb{C}}$. The *Green function* of D is the mapping

$$g_D : D\times D \to (-\infty,+\infty]$$

that satisfies the following conditions for every $w\in D$

(1) $g_D(\cdot,w)$ is harmonic on $D\setminus\{w\}$ and bounded outside every neighbourhood of w,
(2) $g_D(w,w)=\infty$, and for $z\to w$

$$g_D(z,w) = \begin{cases}\log|z| + \mathcal{O}(1),\ w=\infty\\ -\log|z-w| + \mathcal{O}(1),\ w\neq\infty\end{cases}$$

(3) $g_D(z,w)\to 0$, as $z\to\zeta$, for almost every $\zeta\in\partial D$.

Moreover the Green function is symmetric and it holds

$$g_D(z,w) = g_D(w,z),$$

for every $z, w \in D$, when the boundary of D is not polar.

For instance the Green function on the unit disk $\mathbb{D}$ is equal to

$$g_{\mathbb{D}}(z,w) := -\log\tanh d_{\mathbb{D}}(z,w) = \log\left|\frac{1-z\overline{w}}{z-w}\right|, \tag{2.4}$$

for $z, w \in \mathbb{D}$; see [12, p.109].

If the boundary of a domain D is not polar then the Green function g_D exists and it is unique, and the domain D is called *Greenian.*

An important property of the Green function is conformal invariance.

Lemma 2.3 (Subordination Principle)**.** [12, Theorem 4.4.4] *Let D_1, D_2 be two domains of $\widehat{\mathbb{C}}$ with non-polar boundaries and let $f : D_1 \to D_2$ be a meromorphic function. Then*

$$g_{D_1}(z,w) \le g_{D_2}(f(z), f(w)), \quad z, w \in D_1,$$

with equality holding if and only if f is conformal.

Let D be a Greenian domain of $\widehat{\mathbb{C}}$ with Green function $g_D(x,y)$, $x, y \in D$. Suppose E is a compact subset of D. The *Green energy* of E with respect to D is defined as

$$V(E,D) = \inf_{\mu} \int_E \int_E g_D(x,y) d\mu(x) d\mu(y),$$

where the infimum is taken over the Borel measures μ with compact support E and $\mu(E) = 1$. There exists a unique such measure for which this infimum is attained and it is called *Green equilibrium measure.* Moreover, the *Green capacity* of E with respect to D is defined as

$$\operatorname{cap}_D E := \frac{1}{V(E,D)}. \tag{2.5}$$

If E and ∂D have positive logarithmic capacity, the capacity of the condenser (D,E) is proportional to the Green capacity of the compact set E and is given by the formula

$$\operatorname{cap}(D,E) = \frac{2\pi}{V(E,D)} = 2\pi \operatorname{cap}_D E. \tag{2.6}$$

Further information on Green functions and Green capacity can be found in [2], [10], [11] and [12].

Finally, a helpful lemma for the proof of Theorem 1.1 is the following.

Lemma 2.4. [6, Lemma 3.2] *Let D be a Greenian domain in $\widehat{\mathbb{C}}$ and D' a subdomain of D such that the logarithmic capacity* $\operatorname{cap}(D \setminus D') > 0$. *Let K be a compact subset of D' such that* $\operatorname{cap} K > 0$. *Then*

$$\operatorname{cap}(D,K) < \operatorname{cap}(D',K).$$

3 Preliminary lemmas - Proof of Theorem 1.1

Let (ϕ_t) be a semigroup of holomorphic functions in $\mathbb{D}$ with associated Koenigs function h. For the proof of Theorem 1.1, we will need some results from [1].

Lemma 3.1. [1, Proposition 1.4.7] *The semigroup (ϕ_t) is a group of automorphisms of $\mathbb{D}$ if and only if ϕ_{t_0} is an automorphism of $\mathbb{D}$, for some $t_0 \ge 0$.*

Lemma 3.2. [1, Theorem 1.4.22] *The semigroup (ϕ_t) is a group of automorphisms of $\mathbb{D}$ if and only if $h(\mathbb{D})$ is a horizontal half-plane or a horizontal strip.*

Moreover, we need the following monotonicity property of the hyperbolic metric.

Lemma 3.3. [14, Chapter III] *Let z_1, z_2 be two distinct points in $\mathbb{D}$. The hyperbolic distance $d_{\mathbb{D}}(\phi_t(z_1), \phi_t(z_2))$ is a strictly decreasing function of $t \geq 0$, unless (ϕ_t) is a group of automorphisms of the unit disk. In that case, it is constant.*

So, from Lemma 3.3 and (2.4) we obtain the following result.

Corollary 3.1. *Let z_1, z_2 be two distinct points in $\mathbb{D}$. The Green function $g_{\mathbb{D}}(\phi_t(z_1), \phi_t(z_2))$ is a strictly increasing function of $t \geq 0$, unless (ϕ_t) is a group of automorphisms of the unit disk. In that case, it is constant.*

Lemma 3.4. *Consider the semi-infinite strip*

$$L = \{z \in \mathbb{C} : \operatorname{Re} z > 0,\ 0 < \operatorname{Im} z < \pi\}$$

and the horizontal strip

$$S = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}.$$

For $z, w \in L$,

$$g_L(z,w) = g_S(z,w) - g_S(z, -\overline{w}).$$

Proof. The conformal mapping

$$\sigma(z) = \operatorname{Log}\left(\frac{z+i}{z-i}\right)$$

maps the first quadrant $Q = \{z \in \mathbb{C} : 0 < \operatorname{Arg} z < \frac{\pi}{2}\}$ onto the horizontal semi-infinite strip L. Moreover, z^2 maps conformally Q onto the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. For $z, w \in L$, the Green function of L is equal to

$$g_L(z,w) = g_Q\left(\sigma^{-1}(z), \sigma^{-1}(w)\right) = g_{\mathbb{H}}\left(\sigma^{-1}(z)^2, \sigma^{-1}(w)^2\right) = \log\left|\frac{\sigma^{-1}(z)^2 - \overline{\sigma^{-1}(w)}^2}{\sigma^{-1}(z)^2 - \sigma^{-1}(w)^2}\right|, \tag{3.1}$$

see [12, p. 109]. The inverse function of σ is

$$\sigma^{-1}(z) = i\frac{e^z+1}{e^z-1}.$$

With calculations in (3.1), we obtain

$$\begin{aligned} g_L(z,w) &= \log\left|\frac{-\left(\frac{e^z+1}{e^z-1}\right)^2 + \left(\frac{e^{\overline{w}}+1}{e^{\overline{w}}-1}\right)^2}{-\left(\frac{e^z+1}{e^z-1}\right)^2 + \left(\frac{e^{w}+1}{e^{w}-1}\right)^2}\right| \\ &= \log\left|\frac{-(e^z-1)^2(e^w-1)^2}{-(e^z-1)^2(e^{\overline{w}}-1)^2} \cdot \frac{(e^z+1)^2(e^{\overline{w}}-1)^2 - (e^{\overline{w}}+1)^2(e^z-1)^2}{(e^z+1)^2(e^{w}-1)^2 - (e^{w}+1)^2(e^z-1)^2}\right| \\ &= \log\left|\frac{e^z(1-e^{z+\overline{w}}) - e^{\overline{w}}(1-e^{z+\overline{w}})}{e^z(1-e^{z+w}) - e^{w}(1-e^{z+w})}\right| = \log\left|\frac{e^z - e^{\overline{w}}}{e^z - e^w}\right| - \log\left|\frac{e^z - e^{-w}}{e^z - e^{-\overline{w}}}\right| \\ &= g_{\mathbb{H}}(e^z, e^w) - g_{\mathbb{H}}(e^z, e^{-\overline{w}}). \end{aligned}$$

Furthermore, $\operatorname{Log} z$ maps conformally the upper half-plane $\mathbb{H}$ onto the horizontal strip S and we are led to

$$g_L(z,w) = g_S(z,w) - g_S(z, -\overline{w}), \tag{3.2}$$

due to the conformal invariance of the Green function; see [12, Theorem 4.4.4]. □

We will need the following Lemma concerning the asymptotic behavior of the Green function $g_{\mathbb{D}}(\phi_t(z), \phi_t(w))$, for $z, w \in K$.

Lemma 3.5. *Let (ϕ_t) be a one-parameter semigroup with associated Koenigs function h and K be a compact subset of $\mathbb{D}$.*

(i) Suppose (ϕ_t) is a parabolic semigroup of positive hyperbolic step. Then

$$\lim_{t\to+\infty} g_{\mathbb{D}}(\phi_t(\zeta_1),\phi_t(\zeta_2)) = g_H(h(\zeta_1),h(\zeta_2)), \quad \zeta_1,\zeta_2 \in K$$

where H is the smallest horizontal half-plane that contains $h(\mathbb{D})$.

(ii) Suppose (ϕ_t) is a hyperbolic semigroup. Then

$$\lim_{t\to+\infty} g_{\mathbb{D}}(\phi_t(\zeta_1),\phi_t(\zeta_2)) = g_S(h(\zeta_1),h(\zeta_2)), \quad \zeta_1,\zeta_2 \in K$$

where S is the smallest horizontal strip that contains $h(\mathbb{D})$.

Proof. Let $\zeta_1,\zeta_2 \in K$. The limit of $g_{\mathbb{D}}(\phi_t(\zeta_1),\phi_t(\zeta_2))$, as $t\to+\infty$, exists due to Remark 3.1. Set $\Omega = h(\mathbb{D})$. Suppose K has non-zero logarithmic capacity. For $x \in \Omega\cap\mathbb{R}$, define

$$\epsilon(x) = \inf\{y<0 : x+iy \in \Omega\}$$

and

$$M(x) = \sup\{y>0 : x+iy\in\Omega\}.$$

For every $x\in\Omega\cap\mathbb{R}$, we consider the half-strip

$$\Omega_x = \{z\in\Omega : \operatorname{Re} z > x,\ \epsilon(x) < \operatorname{Im} z < M(x)\}.$$

Fix $x\in\Omega\cap\mathbb{R}$. Then for every sufficiently large $t>0$,

$$h(\phi_t(K)) = h(K)+t \subset \Omega_x.$$

The function

$$f(z) = \frac{z}{\pi}(M(x)-\epsilon(x)) + x + i\epsilon(x)$$

maps conformally L onto Ω_x and S onto the horizontal strip

$$S(x) = \{z\in\mathbb{C} : \epsilon(x) < \operatorname{Im} z < M(x)\}.$$

The inverse mapping of f is

$$f^{-1}(z) = \pi\frac{z-x-i\epsilon(x)}{M(x)-\epsilon(x)}.$$

We apply Lemma 3.4 for $\widetilde{z} = f^{-1}(z)$ and $\widetilde{w} = f^{-1}(w)$ and we get

$$g_L(\widetilde{z},\widetilde{w}) = g_L\left(f^{-1}(z), f^{-1}(w)\right) = g_S\left(f^{-1}(z),f^{-1}(w)\right) - g_S\left(f^{-1}(z), -\overline{f^{-1}(w)}\right).$$

We should notice that

$$-\overline{f^{-1}(w)} = \pi\frac{-\overline{w}+x-i\epsilon(x)}{M(x)-\epsilon(x)} = f^{-1}(-\overline{w}+2x)$$

and for $w\in S(x)$, its imaginary part $\operatorname{Im} w = \operatorname{Im}\{-\overline{w}+2x\}$ and so, $-\overline{w}+2x\in S(x)$. Therefore,

$$g_{\Omega_x}(z,w) = g_{S(x)}(z,w) - g_{S(x)}(z,-\overline{w}+2x). \tag{3.3}$$

Suppose $z = h(\phi_t(\zeta_1)), w = h(\phi_t(\zeta_2))$, where $\zeta_1,\zeta_2\in K$. From (3.3), it follows that

$$\begin{aligned} g_\Omega(h(\phi_t(\zeta_1)),h(\phi_t(\zeta_2))) &\geq g_{\Omega_x}(h(\phi_t(\zeta_1)),h(\phi_t(\zeta_2))) = g_{\Omega_x}(h(\zeta_1)+t, h(\zeta_2)+t) \\ &= g_{S(x)}(h(\zeta_1)+t,h(\zeta_2)+t) - g_{S(x)}(h(\zeta_1)+t, -\overline{h(\zeta_2)}-t+2x) \\ &= g_{S(x)}(h(\zeta_1),h(\zeta_2)) - g_{S(x)}(h(\zeta_1)+t,-\overline{h(\zeta_2)}-t+2x), \end{aligned} \tag{3.4}$$

since the Green function of a horizontal strip is invariant under translations parallel to the real axis.

Set U the smallest horizontal domain (half-plane or strip) containing $h(\mathbb{D})$. We obtain for $\zeta_1, \zeta_2 \in K$

$$\begin{aligned} g_U(h(\zeta_1), h(\zeta_2)) &= g_U(h(\zeta_1)+t, h(\zeta_2)+t) \geq g_\Omega(h(\zeta_1)+t, h(\zeta_2)+t) = g_{\mathbb{D}}(\phi_t(\zeta_1), \phi_t(\zeta_2)) \\ &\underset{(3.4)}{\geq} g_{S(x)}(h(\zeta_1), h(\zeta_2)) - g_{S(x)}(h(\zeta_1)+t, -\overline{h(\zeta_2)}-t+2x), \end{aligned} \tag{3.5}$$

Taking the limit as $t \to +\infty$ in (3.5), we have

$$\begin{aligned} g_U(h(\zeta_1), h(\zeta_2)) &\geq \lim_{t\to+\infty} g_{\mathbb{D}}(\phi_t(\zeta_1), \phi_t(\zeta_2)) \\ &\geq g_{S(x)}(h(\zeta_1), h(\zeta_2)) - \lim_{t\to+\infty} g_{S(x)}(h(\zeta_1)+t, -\overline{h(\zeta_2)}-t+2x), \end{aligned} \tag{3.6}$$

for every $\zeta_1, \zeta_2 \in K$. However, it is clear that

$$\lim_{t\to+\infty} g_{S(x)}(h(\zeta_1)+t, -\overline{h(\zeta_2)}-t+2x) = 0.$$

As a result, (3.6) can be written as

$$g_U(h(\zeta_1), h(\zeta_2)) \geq \lim_{t\to+\infty} g_{\mathbb{D}}(\phi_t(\zeta_1), \phi_t(\zeta_2)) \geq g_{S(x)}(h(\zeta_1), h(\zeta_2)), \tag{3.7}$$

for every $\zeta_1, \zeta_2 \in K$. Since (3.7) holds for sufficiently large x, it is true that

$$g_U(h(\zeta_1), h(\zeta_2)) \geq \lim_{t\to+\infty} g_{\mathbb{D}}(\phi_t(\zeta_1), \phi_t(\zeta_2)) \geq \lim_{x\to+\infty} g_{S(x)}(h(\zeta_1), h(\zeta_2)),$$

for every $\zeta_1, \zeta_2 \in K$.

Suppose that the semigroup (ϕ_t) is hyperbolic. Then the domain Ω is contained in a horizontal strip; see [7, Theorem 2.1]. Let $S = \{z \in \mathbb{C} : \rho_1 < \operatorname{Im} z < \rho_2\}$, $\rho_1 < 0 < \rho_2$, be the smallest horizontal strip that contains Ω. If $x \to +\infty$, then $\epsilon(x) \to \rho_1$ and $M(x) \to \rho_2$. Thus $\{S(x)\}_x$ is an increasing sequence of domains converging to S. So, the limit

$$\lim_{x\to+\infty} g_{S(x)}(h(\zeta_1), h(\zeta_2)) = g_S(h(\zeta_1), h(\zeta_2)), \quad \zeta_1, \zeta_2 \in K$$

due to the property of domain monotonicity of the Green function. Therefore, we have that

$$\lim_{t\to+\infty} g_{\mathbb{D}}(\phi_t(\zeta_1), \phi_t(\zeta_2)) = g_S(h(\zeta_1), h(\zeta_2)),$$

for any $\zeta_1, \zeta_2 \in K$.

In the case, where (ϕ_t) is a parabolic semigroup of positive hyperbolic step, the proof is similar. The domain Ω is contained in a horizontal half-plane; see [7, Theorem 2.1].

Let $H = \{z \in \mathbb{C} : \operatorname{Im} z > -\rho\}$, $\rho > 0$, be the smallest horizontal half-plane that contains Ω. If $x \to +\infty$, then $\epsilon(x) \to \rho$ and $M(x) \to +\infty$. Thus $\{S(x)\}_x$ is an increasing sequence of domains converging to H. So, the limit

$$\lim_{x\to+\infty} g_{S(x)}(h(\zeta_1), h(\zeta_2)) = g_H(h(\zeta_1), h(\zeta_2)), \quad \zeta_1, \zeta_2 \in K$$

due to the property of domain monotonicity of the Green function. Consequently, the limit

$$\lim_{t\to+\infty} g_{\mathbb{D}}(\phi_t(\zeta_1), \phi_t(\zeta_2)) = g_H(h(\zeta_1), h(\zeta_2)),$$

for any $\zeta_1, \zeta_2 \in K$. □

Proof of Theorem 1.1. Suppose first that $\phi_t \notin \operatorname{Aut}(\mathbb{D})$, for any t. Let $s > 0$. Suppose K is a compact subset of the unit disk $\mathbb{D}$. Since every function ϕ_t of the semigroup is univalent, it holds

$$\operatorname{cap}(\mathbb{D}, \phi_t(K)) = \operatorname{cap}(\phi_s(\mathbb{D}), \phi_s(\phi_t(K))) = \operatorname{cap}(\phi_s(\mathbb{D}), \phi_{t+s}(K)), \tag{3.8}$$

due to conformal invariance.

By Lemma 2.1,

$$\operatorname{cap}(\mathbb{D}, \phi_{t+s}(K)) \leq \operatorname{cap}(\phi_s(\mathbb{D}), \phi_{t+s}(K)). \tag{3.9}$$

Since none of the functions ϕ_t is an automorphism, there exists a point $\zeta \in \mathbb{D} \cap \partial\phi_s(\mathbb{D})$. Therefore, there exists a continuum Γ that joins this point ζ with the unit circle, which has positive logarithmic capacity.

Then from Lemma 2.4, we obtain

$$\operatorname{cap}(\mathbb{D}, \phi_{t+s}(K)) < \operatorname{cap}(\mathbb{D} \setminus \Gamma, \phi_{t+s}(K)). \tag{3.10}$$

By the domain monotonicity and Lemma 2.1,

$$\operatorname{cap}(\mathbb{D} \setminus \Gamma, \phi_{t+s}(K)) \leq \operatorname{cap}(\phi_s(\mathbb{D}), \phi_{t+s}(K)). \tag{3.11}$$

Combining inequalities (3.10) and (3.11) with (3.8), we conclude that

$$\operatorname{cap}(\mathbb{D}, \phi_{t+s}(K)) < \operatorname{cap}(\mathbb{D}, \phi_t(K))$$

and so, $\operatorname{cap}(\mathbb{D}, \phi_t(K))$ is a strictly decreasing function of t.

If $\phi_{t_0} \in \operatorname{Aut}(\mathbb{D})$, for some $t_0 > 0$, then equality is attained in (3.9) and due to (3.8), the capacity $\operatorname{cap}(\mathbb{D}, \phi_t(K))$ is a constant function of t. □

4 Proof of Theorem 1.2

Suppose that (ϕ_t) is a hyperbolic or a parabolic semigroup of positive hyperbolic step with associated Koenigs function h. Recall from Section 3 that for $x \in \Omega \cap \mathbb{R}$, we have defined the horizontal strip

$$S(x) = \{z \in \mathbb{C} : \epsilon(x) < \operatorname{Im} z < M(x)\}.$$

We obtain the following lemma.

Lemma 4.1. *Let K be a compact subset of $\mathbb{D}$. Then*

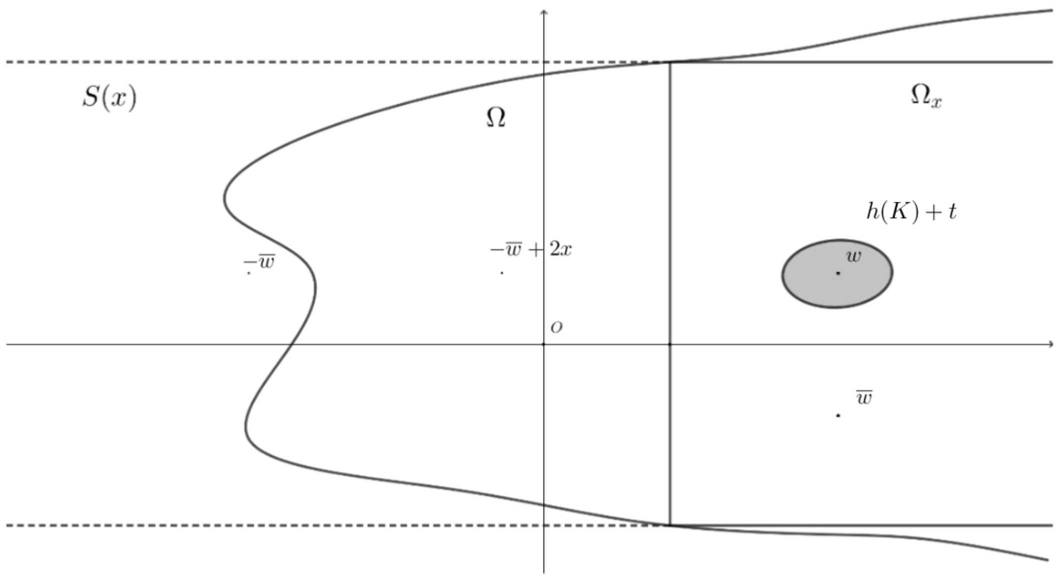
$$\lim_{t \to +\infty} \operatorname{cap}(\mathbb{D}, \phi_t(K)) \leq \operatorname{cap}(S(x), h(K)). \tag{4.1}$$

Proof. Let μ_t be the Green equilibrium measure on $h(K) + t$ with respect to Ω. It follows from (3.4) that

$$\begin{aligned}
\int\int_{(h(K)+t)^2} g_\Omega(z,w)d\mu_t(z)d\mu_t(w) &\geq \int\int_{(h(K)+t)^2} g_{S(x)}(z,w)d\mu_t(z)d\mu_t(w) \\
&\quad - \int\int_{(h(K)+t)^2} g_{S(x)}(z,-\overline{w}+2x)d\mu_t(z)d\mu_t(w) \\
&\geq \int\int_{(h(K)+t)^2} g_{S(x)}(z,w)d\mu_t^\star(z)d\mu_t^\star(w) \\
&\quad - \int\int_{(h(K)+t)^2} g_{S(x)}(z,-\overline{w}+2x)d\mu_t(z)d\mu_t(w),
\end{aligned} \tag{4.2}$$

where $\mu_t^\star$ is the Green equilibrium measure on $h(K) + t$ with respect to $S(x)$. For $w \in h(K) + t$, the point $-\overline{w} + 2x$ is not contained in $h(K) + t$, because $\operatorname{Im}\{-\overline{w} + 2x\} = \operatorname{Im} w$ but

$$\operatorname{Re}\{-\overline{w} + 2x\} = -\operatorname{Re} w + 2x = x - (\operatorname{Re} w - x) < x.$$



So, the function $g_{S(x)}(\cdot,-\overline{w}+2x)$ is harmonic and bounded on $h(K)+t$; see [12, Definition 4.4.1]. Therefore, it satisfies the Maximum Principle and there exists a point $z_1 \in \partial h(K)+t$ such as

$$g_{S(x)}(z_1,-\overline{w}+2x) = \max_{z\in h(K)+t} g_{S(x)}(z,-\overline{w}+2x), \quad w \in h(K)+t.$$

Due to the fact that the Green function is symmetric [12, Theorem 4.4.8], $g_{S(x)}(z_1,\cdot)$ is harmonic and bounded on the set $\{-\overline{w}+2x : w \in h(K)+t\}$. Hence, again from the Maximum Principle, there exists a point $w_1 \in \partial h(K)+t$ such that

$$g_{S(x)}(z_1,-\overline{w_1}+2x) = \max_{w\in h(K)+t} g_{S(x)}(z_1,-\overline{w}+2x).$$

The points $z_1, w_1 \in h(K)+t$ and so, we can write that $z_1 = h(\zeta_1)+t$, $w_1 = h(\zeta_2)+t$ and

$$g_{S(x)}(z_1,-\overline{w_1}+2x) = g_{S(x)}(h(\zeta_1)+t,-\overline{h(\zeta_2)}-t+2x), \quad \zeta_1,\zeta_2 \in K.$$

As a result,

$$\int\int_{h(K)+t)^2} g_{S(x)}(z,-\overline{w}+2x)d\mu_t(z)d\mu_t(w) \le \mu_t(h(K)+t)^2 g_{S(x)}(z_1,-\overline{w_1}+2x) = g_{S(x)}(z_1,-\overline{w_1}+2x)$$

and (4.2) becomes

$$\int\int_{(h(K)+t)^2} g_\Omega(z,w)d\mu_t(z)d\mu_t(w) \ge \int\int_{(h(K)+t)^2} g_{S(x)}(z,w)d\mu_t^\star(z)d\mu_t^\star(w) - g_{S(x)}(z_1,-\overline{w}_1+2x).$$

Taking the limit as $t\to+\infty$, we have that

$$\begin{aligned}\lim_{t\to+\infty}\int\int_{(h(K)+t)^2} g_\Omega(z,w)d\mu_t(z)d\mu_t(w) \ge& \lim_{t\to+\infty}\int\int_{(h(K)+t)^2} g_{S(x)}(z,w)d\mu_t^\star(z)d\mu_t^\star(w)\\ &- \lim_{t\to+\infty} g_{S(x)}(h(\zeta_1)+t,-\overline{h(\zeta_2)}-t+2x)\\ =& \lim_{t\to+\infty}\int\int_{(h(K)+t)^2} g_{S(x)}(z,w)d\mu_t^\star(z)d\mu_t^\star(w).\end{aligned}$$

Using the Green capacity of $h(K)+t$ with respect to Ω and the strip $S(x)$ (see (2.5)), respectively, the above inequality can be written as

$$\lim_{t\to+\infty} \operatorname{cap}_\Omega(h(K)+t) \leq \lim_{t\to+\infty} \operatorname{cap}_{S(x)}(h(K)+t) = \operatorname{cap}_{S(x)} h(K),$$

since the Green capacity with respect to a horizontal strip is invariant under translations parallel to the real axis. Let's also recall that $\operatorname{cap}_{\mathbb{D}} \phi_t(K) = \operatorname{cap}_\Omega(h(K)+t)$, due to the conformal invariance of the Green function. Consequently,

$$\lim_{t\to+\infty} \operatorname{cap}_{\mathbb{D}} \phi_t(K) \leq \operatorname{cap}_{S(x)} h(K)$$

or equivalently, using condenser capacity

$$\lim_{t\to+\infty} \operatorname{cap}(\mathbb{D}, \phi_t(K)) \leq \operatorname{cap}(S(x), h(K)), \tag{4.3}$$

for every $x > 0$. □

This result will also be needed at the proof of Theorem 1.3.

Completion of proof of Theorem 1.2. Suppose that the semigroup (ϕ_t) is hyperbolic. Let $S_{\rho_1,\rho_2} = \{z \in \mathbb{C} : \rho_1 < \operatorname{Im} z < \rho_2\}$ be the smallest horizontal strip that contains Ω. Then, due to Lemma 2.1, we have that

$$\operatorname{cap}(\mathbb{D}, \phi_t(K)) = \operatorname{cap}(\Omega, h(K)+t) \geq \operatorname{cap}(S_{\rho_1,\rho_2}, h(K)+t) = \operatorname{cap}(S_{\rho_1,\rho_2}, h(K)).$$

With the use of (4.1), we obtain the following inequality

$$\operatorname{cap}(S_{\rho_1,\rho_2}, h(K)) \leq \lim_{t\to+\infty} \operatorname{cap}(\mathbb{D}, \phi_t(K)) \leq \operatorname{cap}(S(x), h(K)). \tag{4.4}$$

Let $x \to +\infty$. Then $\epsilon(x) \to \rho_1$ and $M(x) \to \rho_2$. Therefore, the condensers $(S(x), h(K))$ form an exhaustion of the condenser $(S_{\rho_1,\rho_2}, h(K))$ and according to Lemma 2.2,

$$\lim_{x\to+\infty} \operatorname{cap}(S(x), h(K)) = \operatorname{cap}(S_{\rho_1,\rho_2}, h(K)). \tag{4.5}$$

As a result, taking the limit as $x \to +\infty$ in (4.4), and using (4.5), we find that

$$\lim_{t\to+\infty} \operatorname{cap}(\mathbb{D}, \phi_t(K)) = \operatorname{cap}(S_{\rho_1,\rho_2}, h(K)).$$

□

5 Proof of Theorem 1.3

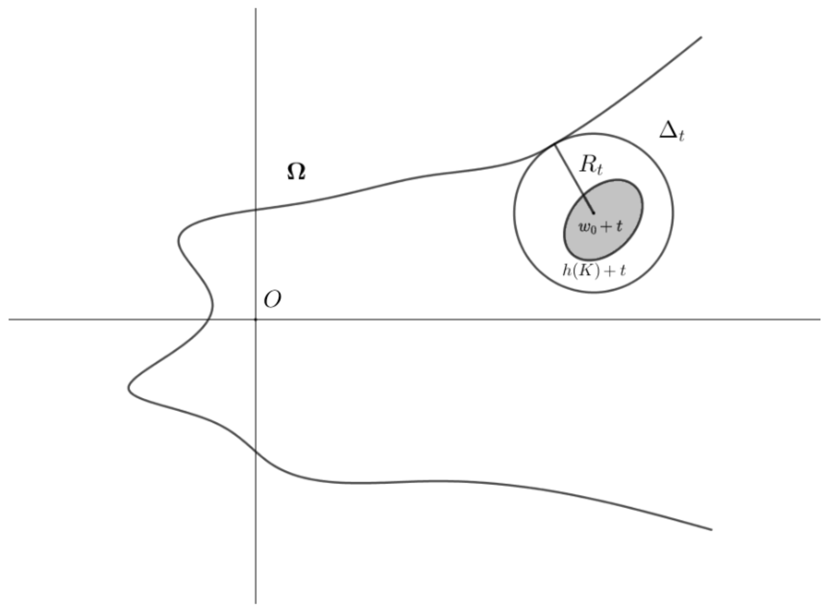
Suppose first that (ϕ_t) is a parabolic semigroup of zero hyperbolic step. According to [5, Corollary 1], for every $w \in \Omega$ it holds

$$\lim_{t\to+\infty} \operatorname{dist}(w+t, \partial\Omega) = +\infty. \tag{5.1}$$

Fix $w_0 \in h(K)$. For every $t > 0$, let

$$R_t := \operatorname{dist}(w_0+t, \partial\Omega),$$

and due to (5.1), $R_t \xrightarrow{t\to+\infty} +\infty$. Then $h(K)+t \subset \Delta_t := D(w_0+t, R_t) \subset \Omega$, for sufficiently large t.



Hence

$$\begin{aligned}\operatorname{cap}(\mathbb{D},\phi_t(K)) &= \operatorname{cap}(\Omega,h(K)+t) \le \operatorname{cap}(\Delta_t,h(K)+t)\\ &= \operatorname{cap}(D(w_0,R_t),h(K)) \le \operatorname{cap}(D(w_0,R_t),\overline{D(w_0,\operatorname{diam}h(K))})\\ &= 2\pi\left(\log\frac{R_t}{\operatorname{diam}h(K)}\right)^{-1} \xrightarrow{t\to+\infty} 0.\end{aligned}$$

Next we examine the case where (ϕ_t) is a parabolic semigroup of positive hyperbolic step. The domain $\Omega = h(\mathbb{D})$ is contained in a horizontal half-plane; see e.g. [5, Theorem 1]. Let $H_\rho = \{z \in \mathbb{C} : \operatorname{Im} z > -\rho\}$, $\rho > 0$, be the smallest such half-plane. Then, due to Lemma 2.1, the following inequality is true

$$\operatorname{cap}(\mathbb{D},\phi_t(K)) = \operatorname{cap}(\Omega,h(K)+t) \ge \operatorname{cap}(H_\rho,h(K)+t) = \operatorname{cap}(H_\rho,h(K)). \tag{5.2}$$

According to Section 4 and Lemma 4.1,

$$\lim_{t\to+\infty}\operatorname{cap}(\mathbb{D},\phi_t(K)) \le \operatorname{cap}(S(x),h(K)),$$

where $S(x) = \{z\in\mathbb{C} : \epsilon(x) < \operatorname{Im} z < M(x)\}$. Hence, (5.2) gives

$$\operatorname{cap}(H_\rho,h(K)) \le \lim_{t\to+\infty}\operatorname{cap}(\mathbb{D},\phi_t(K)) \le \operatorname{cap}(S(x),h(K)). \tag{5.3}$$

Taking the limit $x\to+\infty$, we have that $\epsilon(x)$ tends to $-\rho$, whereas, $M(x)$ tends to $+\infty$. The condensers $(S(x),h(K))$ form an exhaustion of the condenser $(H_\rho,h(K))$ and according to Lemma 2.2,

$$\lim_{x\to+\infty}\operatorname{cap}(S(x),h(K)) = \operatorname{cap}(H_\rho,h(K)). \tag{5.4}$$

Therefore, taking the limit $x\to+\infty$ in (5.3), we obtain

$$\lim_{t\to+\infty}\operatorname{cap}(\mathbb{D},\phi_t(K)) = \operatorname{cap}(H_\rho,h(K)).$$

□

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