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Strong Ambiguity

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We examine the conditions under which a model of Tangled Type Theory satisfies the same sentences as a model of NF (assuming we ignore type indices).

1 Introduction

Tangled Type Theory (TTT) was introduced by Holmes in [2] as a modified version of Simple Type Theory (TST) which is equiconsistent with Quine's "New Foundations" (NF). In this paper, we present conditions under which a model of TTT is essentially equivalent to a model of NF. We first show that every sentence of the language of NF can be expressed as a sentence of the language of TTT where all type indices have been erased (we call such sentences semistratified). We then introduce the notion of tangled type-shifting automorphism for structures of the language of TTT, and show that every structure that has such an automorphism satisfies the same sentences as a structure of the language of NF (assuming we ignore type indices). We also prove that the existence of a tangled type-shifting automorphism is equivalent to a strengthening of the Axiom scheme of Ambiguity which we call the Axiom scheme of Strong Ambiguity. This new axiom scheme basically asserts that the truth value of a sentence is preserved when we raise the type of one of its variables. Finally, we define the notion of permutation model in the context of TTT, and use it to prove the independence of the Axiom scheme of Strong Ambiguity.

We briefly review some of the basic notions that we will use in this paper. We note that our metatheory throughout this paper will be ZF with $\in$ as the membership relation.

1.1 Simple Type Theory

The language of Simple Type Theory ($\mathcal{L}_{\mathsf{TST}}$) is the many-sorted language of set theory with one binary relation symbol ε and countably many types indexed by ω. For each type $i \in \omega$, there are countably many variables $v^i_0, v^i_1, \dots$ of type i (we use a simpler notation like $x^i, y^i, u^i, v^i, w^i, \dots$ to refer to these variables). The $\mathcal{L}_{\mathsf{TST}}$-formulas are built inductively from the atomic formulas $x^i \,\varepsilon\, y^{i+1}$ and $x^i = y^i$ in the usual way.

Simple Type Theory (TST) is axiomatized by two sets of axioms. The *Axiom of Extensionality* (Ext) is the set of all the following sentences for each type $i \in \omega$,

$$\forall x^{i+1}, y^{i+1}(x^{i+1} = y^{i+1} \leftrightarrow \forall z^i(z^i \,\varepsilon\, x^{i+1} \leftrightarrow z^i \,\varepsilon\, y^{i+1})). \qquad (\mathsf{Ext}^{i+1})$$

The *Axiom scheme of Comprehension* (Co) is the set of all the following sentences for each type $i \in \omega$ and formula φ of $\mathcal{L}_{\mathsf{TST}}$,

$$\forall \bar{u} \exists y^{i+1} \forall x^i (x^i \,\varepsilon\, y^{i+1} \leftrightarrow \varphi(x^i, \bar{u})), \qquad (\mathsf{Co}^{i+1})$$

where y^{i+1} is not free in φ.

The language of Tangled Type Theory ($\mathcal{L}_{\mathsf{TTT}}$) is the same as $\mathcal{L}_{\mathsf{TST}}$, but its formulas are built inductively from the atomic formulas $x^i = y^i$ and $x^i \,\varepsilon\, y^j$ for $i < j < \omega$ (notice that every formula of $\mathcal{L}_{\mathsf{TST}}$ is also a formula of $\mathcal{L}_{\mathsf{TTT}}$). For each function $s : \omega \to \omega$ and each $\mathcal{L}_{\mathsf{TST}}$-formula φ, we denote by φ^s the $\mathcal{L}_{\mathsf{TTT}}$-formula we get if we replace

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each type index i of a variable in φ with $s(i)$ (i.e., if we replace each v_k^i with $v_k^{s(i)}$). *Tangled Type Theory* (TTT) is defined as

$$\mathsf{TTT} = \{\sigma^s : \sigma \in \mathsf{TST} \text{ and } s : \omega \to \omega \text{ strictly increasing}\}.$$

A *structure* $\mathscr{A}$ for the language $\mathcal{L}_{\mathsf{TST}}$ is a sequence $(A_0, A_1, \ldots, \{\varepsilon_{i,i+1}^{\mathscr{A}}\}_{i\in\omega})$, where $A_0, A_1, \ldots$ are non-empty sets interpreting the ω types of $\mathcal{L}_{\mathsf{TST}}$, and each $\varepsilon_{i,i+1}^{\mathscr{A}} \subseteq A_i \times A_{i+1}$ is a binary relation interpreting ε for type $i \in \omega$. Similarly, a *structure* $\mathscr{A}$ for the language $\mathcal{L}_{\mathsf{TTT}}$ is a sequence $(A_0, A_1, \ldots, \{\varepsilon_{i,j}^{\mathscr{A}})\}_{i<j})$, where $A_0, A_1, \ldots$ are non-empty sets, and $\varepsilon_{i,j}^{\mathscr{A}} \subseteq A_i \times A_j$, for all $i < j < \omega$.

1.2 New Foundations

The language $\mathcal{L}_{\mathsf{NF}}$ of New Foundations is the usual one-sorted language of set theory, $\{\varepsilon\}$, where ε is a binary relation symbol. To every formula φ of $\mathcal{L}_{\mathsf{TTT}}$ we may assign a unique formula φ^* of $\mathcal{L}_{\mathsf{NF}}$ which can be obtained by erasing all type indices from the variables of φ. A formula φ of $\mathcal{L}_{\mathsf{NF}}$ is *stratified* if there exists an $\mathcal{L}_{\mathsf{TST}}$-formula ψ such that $\varphi = \psi^*$. If Γ is a set of $\mathcal{L}_{\mathsf{TTT}}$-sentences, then we let $\Gamma^* = \{\sigma^* : \sigma \in \Gamma\}$. *New Foundations* (NF) is axiomatized by the *Axiom of Extensionality* (Ext),

$$\forall x, y(x = y \leftrightarrow \forall z(z \,\varepsilon\, x \leftrightarrow z \,\varepsilon\, y)), \tag{Ext}$$

and the *Axiom scheme of Stratified Comprehension* (Co), which consists of all sentences

$$\forall \bar{u} \exists y \forall x(x \,\varepsilon\, y \leftrightarrow \varphi(x, \bar{u})), \tag{Co}$$

where φ is a stratified $\mathcal{L}_{\mathsf{NF}}$-formula such that y is not free in φ. Notice that $\mathsf{NF} = \mathsf{TST}^* = \mathsf{TTT}^*$.

Specker established a very important relation between NF and TST (cf. [6] or [1]). The *Axiom scheme of Ambiguity* (Amb) is the set of all $\sigma \leftrightarrow \sigma^+$, where σ is an $\mathcal{L}_{\mathsf{TST}}$-sentence, and σ^+ is the $\mathcal{L}_{\mathsf{TST}}$-sentence we get from σ if we raise the type of each variable by one (i.e., the sentence we get if we replace each v_k^i with v_k^{i+1}). If $\mathscr{A} = (A_0, A_1, \ldots, \{\varepsilon_{i,i+1}^{\mathscr{A}}\}_{i\in\omega})$ is an $\mathcal{L}_{\mathsf{TST}}$-structure, then a sequence $f = (f_0, f_1, \ldots)$ is called a *type-shifting automorphism of* $\mathscr{A}$ (or just a *tsau*) if

(i) for all $i \in \omega$, $f_i : A_i \to A_{i+1}$ is a bijection, and
(ii) for all $i \in \omega$, $x \in A_i$, and $y \in A_{i+1}$,

$$x \,\varepsilon_{i,i+1}^{\mathscr{A}}\, y \iff f_i(x) \,\varepsilon_{i+1,i+2}^{\mathscr{A}}\, f_{i+1}(y).$$

Theorem 1.1 (Specker) *If $\mathcal{M}$ is a model of* $\mathsf{TST} + \mathsf{Amb}$*, then there exists an $\mathcal{M}'$ elementarily equivalent to $\mathcal{M}$ with a tsau.*

NF is strongly connected to TTT as well. In particular, Holmes showed that these two theories are equiconsistent (cf. [2]). His proof is based on a modification of the argument used by Jensen to prove the consistency of NF with urelements (cf. [4]).

Theorem 1.2 (Holmes) NF *is consistent if and only if* TTT *is consistent.*

Note. It should also be noted that TTT is crucial in Holmes' proof of the consistency of NF (cf. [3]), a proof that is still unverified though.

Below, we investigate further how deep this connection between TTT and NF really is.

2 Strong Ambiguity

First of all, let us show how any unstratified $\mathcal{L}_{\mathsf{NF}}$-sentence can be expressed as an $\mathcal{L}_{\mathsf{TTT}}$-sentence (assuming we ignore type indices).

Definition 2.1 An $\mathcal{L}_{\mathsf{NF}}$-formula φ is called *semistratified* if it is ψ^* for some $\mathcal{L}_{\mathsf{TTT}}$-formula ψ.

Obviously, every stratified $\mathcal{L}_{\mathsf{NF}}$-formula is semistratified, but semistratified formulas can express much more.

Theorem 2.2 *For every $\mathcal{L}_{\mathsf{NF}}$-formula φ, there exists a semistratified $\mathcal{L}_{\mathsf{NF}}$-formula ψ such that* $\mathsf{Ext} \vdash \varphi \leftrightarrow \psi$.

Proof. By induction on $\mathcal{L}_{\mathsf{NF}}$-formula φ. We only need to check the cases where φ is unstratified. If φ is $x \,\varepsilon\, x$, then $\mathsf{Ext} \vdash \varphi \leftrightarrow \psi^*$, where ψ is the $\mathcal{L}_{\mathsf{TTT}}$-formula $\exists z^2(\forall v^0(v^0 \,\varepsilon\, z^2 \leftrightarrow v^0 \,\varepsilon\, x^1) \wedge x^1 \,\varepsilon\, z^2))$. The cases where φ is $\neg\psi$ or $\exists x\psi$, for some $\mathcal{L}_{\mathsf{NF}}$-formula ψ, are trivial. Assume now that φ is $\mathrm{Q}_1\, x_1 \dots \mathrm{Q}_n\, x_n \varphi_1 \wedge \mathrm{Q}'_1\, y_1 \dots \mathrm{Q}'_m\, y_m \varphi_2$, for some $\mathcal{L}_{\mathsf{NF}}$-formulas $\varphi_1(x_1, \dots, x_n, u_1, \dots, u_k)$ and $\varphi_2(y_1, \dots, y_m, u_1, \dots, u_k)$. We may assume that the variables $x_1, \dots, x_n, y_1, \dots, y_m$ are distinct. By the induction hypothesis, there are $\mathcal{L}_{\mathsf{TTT}}$-formulas $\psi_1(x_1^{i_1}, \dots, x_n^{i_n}, u_1^{s_1}, \dots, u_k^{s_k})$ and $\psi_2(y_1^{j_1}, \dots, y_m^{j_m}, u_1^{r_1}, \dots, u_k^{r_k})$ such that φ_1 is ψ_1^* and φ_2 is ψ_2^*. Let χ be the following $\mathcal{L}_{\mathsf{TTT}}$-formula,

$$\begin{aligned}
&\mathrm{Q}_1\, x_1^{i_1+1} \dots \mathrm{Q}_n\, x_n^{i_n+1} \psi_1(x_1^{i_1+1}, \dots, x_n^{i_n+1}, u_1^{s_1+1}, \dots, u_k^{s_k+1}) \\
&\wedge \exists w_1^{r_1+1}, \dots, w_k^{r_k+1} \Big(\bigwedge_{1 \le i \le k} (\forall v^0 (v^0 \,\varepsilon\, u_i^{s_i+1} \leftrightarrow v^0 \,\varepsilon\, w_i^{r_i+1})) \\
&\qquad \wedge \mathrm{Q}'_1\, y_1^{j_1+1} \dots \mathrm{Q}'_m\, y_m^{j_m+1} \psi_2(y_1^{j_1+1}, \dots, y_m^{j_m+1}, w_1^{r_1+1}, \dots, w_k^{r_k+1}) \Big).
\end{aligned}$$

It is easy to see that $\mathsf{Ext} \vdash \varphi \leftrightarrow \chi^*$. □

Let us now introduce the notion of tangled type-shifting automorphism for $\mathcal{L}_{\mathsf{TTT}}$-structures. We will show that an $\mathcal{L}_{\mathsf{TTT}}$-structure with a tangled type-shifting automorphism is basically a "flattened" version of some $\mathcal{L}_{\mathsf{NF}}$-structure, i.e., the two structures satisfy the same $\mathcal{L}_{\mathsf{TTT}}$-sentences if we ignore type indices.

Definition 2.3 Let $\mathscr{A} = (A_0, A_1, \dots, \{\varepsilon_{i,j}^{\mathscr{A}}\}_{i<j})$ be an $\mathcal{L}_{\mathsf{TTT}}$-structure. Let f be a sequence of functions $(f_0, f_1, \dots)$. We say that f is a *tangled type-shifting automorphism (or just a tangled tsau) of $\mathscr{A}$* if

(i) for all $i \in \omega$, $f_i : A_i \to A_{i+1}$ is a bijection, and
(ii) for all $i < j < \omega$ and $k < l < \omega$, where $i \le k$ and $j \le l$, and for all $x \in A_i$ and $y \in A_j$,

$$x \,\varepsilon_{i,j}^{\mathscr{A}}\, y \iff f_{k-1} \circ \dots \circ f_i(x) \,\varepsilon_{k,l}^{\mathscr{A}}\, f_{l-1} \circ \dots \circ f_j(y).$$

Note. We should clarify that $f_{j-1} \circ \dots \circ f_i$ denotes the identity when $j = i$. So, for example, condition (ii) above implies that

$$x \,\varepsilon_{i,i+2}^{\mathscr{A}}\, y \iff x \,\varepsilon_{i,i+3}^{\mathscr{A}}\, f_{i+2}(y) \iff f_i(x) \,\varepsilon_{i+1,i+2}^{\mathscr{A}}\, y.$$

Lemma 2.4 *Let $\mathscr{A} = (A_0, A_1, \dots, \{\varepsilon_{i,j}^{\mathscr{A}}\}_{i<j})$ be an $\mathcal{L}_{\mathsf{TTT}}$-structure with a tangled tsau $f = (f_0, f_1, \dots)$. Let $\mathcal{M} = (A_0, \varepsilon^{\mathcal{M}})$ be the $\mathcal{L}_{\mathsf{NF}}$-structure where for all $x, y \in A_0$, $x \,\varepsilon^{\mathcal{M}}\, y$ iff $x \,\varepsilon_{0,1}^{\mathscr{A}}\, f_0(y)$. For any $a_1, \dots, a_n \in A_0$ and $\mathcal{L}_{\mathsf{TTT}}$-formula $\varphi(x_1^{i_1}, \dots, x_n^{i_n})$,*

$$\mathcal{M} \models \varphi^*(a_1, \dots, a_n) \iff \mathscr{A} \models \varphi(f_{i_1-1} \circ \dots \circ f_0(a_1), \dots, f_{i_n-1} \circ \dots \circ f_0(a_n)).$$

In particular, $\mathcal{M} \models \mathsf{NF}$ iff $\mathscr{A} \models \mathsf{TTT}$.

Proof. By induction on $\mathcal{L}_{\mathsf{TTT}}$-formula φ. If φ is $x^i = x^i$, then since $f_{i-1} \circ \dots \circ f_0$ is a bijection, we have that for all $a_1, a_2 \in A_0$, $a_1 = a_2$ iff $f_{i-1} \circ \dots \circ f_0(a_1) = f_{i-1} \circ \dots \circ f_0(a_2)$. If φ is $x_1^{i_1} \,\varepsilon\, x_2^{i_2}$, then for all $a_1, a_2 \in A_0$, $a_1 \,\varepsilon^{\mathcal{M}}\, a_2$ is equivalent to $a_1 \,\varepsilon_{0,1}^{\mathscr{A}}\, f_0(a_2)$, which by the definition of a tangled tsau is equivalent to $f_{i_1-1} \circ \dots \circ f_0(a_1) \,\varepsilon_{i_1,i_2}^{\mathscr{A}}\, f_{i_2-1} \circ \dots \circ f_0(a_2)$. The cases where φ is $\neg\psi$ or $\psi \wedge \chi$, for some $\mathcal{L}_{\mathsf{TTT}}$-formulas ψ and χ, are trivial. Assume now that φ is $\exists x^i \psi(x^i, x_1^{i_1}, \dots, x_n^{i_n})$, for some $\mathcal{L}_{\mathsf{TTT}}$-formula ψ. We have $\mathcal{M} \models \varphi^*(a_1, \dots, a_n)$ iff $\mathcal{M} \models \exists x \psi^*(x, a_1, \dots, a_n)$, i.e., iff there is an $x \in A_0$ such that $\mathcal{M} \models \psi^*(x, a_1, \dots, a_n)$, which by the induction hypothesis holds iff there is an $x \in A_0$ such that $\mathscr{A} \models \psi(f_{i-1} \circ \dots \circ f_0(x), f_{i_1-1} \circ \dots \circ f_0(a_1), \dots, f_{i_n-1} \circ \dots \circ f_0(a_n))$, which in turn holds iff there is an $x \in A_i$ such that $\mathscr{A} \models \psi(x, f_{i_1-1} \circ \dots \circ f_0(a_1), \dots, f_{i_n-1} \circ \dots \circ f_0(a_n))$ (because $f_{i-1} \circ \dots \circ f_0$ is onto), which is equivalent to $\mathscr{A} \models \exists x^i \psi(x^i, f_{i_1-1} \circ \dots \circ f_0(a_1), \dots, f_{i_n-1} \circ \dots \circ f_0(a_n))$. □

As we will see, the existence of a tangled tsau is equivalent to a strengthening of the Axiom scheme of Ambiguity.

Definition 2.5 Let σ be an $\mathcal{L}_{\mathsf{TTT}}$-sentence and x^i be a variable. If σ satisfies the following two conditions,

(i) the variable x^{i+1} does not appear in σ (i.e., if x^i is v_k^i, then v_k^{i+1} does not appear in σ), and
(ii) for all variables y^i, w^{i+1}, the atomic formulas $x^i = y^i$, $y^i = x^i$, and $x^i \mathrel{\varepsilon} w^{i+1}$ do not appear in σ,

then let σ^{x^i+} be the $\mathcal{L}_{\mathsf{TTT}}$-sentence we get from σ if we replace each occurrence of x^i with x^{i+1}. If σ does not satisfy the above conditions, then let σ^{x^i+} be σ.

The *Axiom scheme of Strong Ambiguity* (StAmb) is defined to be the set of all $\sigma \leftrightarrow \sigma^{x^i+}$, where σ is an $\mathcal{L}_{\mathsf{TTT}}$-sentence σ and x^i is a variable.

The following proposition confirms that Strong Ambiguity is indeed stronger than Ambiguity.

Proposition 2.6 *If T is an $\mathcal{L}_{\mathsf{TTT}}$-theory such that for all $i \in \omega$, $T \vdash \forall x^i, y^i(\forall w^{i+1}(x^i \mathrel{\varepsilon} w^{i+1} \leftrightarrow y^i \mathrel{\varepsilon} w^{i+1}) \rightarrow x^i = y^i)$ (note that TTT is clearly such a theory), then $T \vdash \mathsf{StAmb} \rightarrow \mathsf{Amb}$.*

Proof. Let T be such a theory, and let σ be an $\mathcal{L}_{\mathsf{TST}}$-sentence. Let τ be the $\mathcal{L}_{\mathsf{TST}}$-sentence we get from σ if for all variables x^i, y^i, we replace each occurrence of $x^i = y^i$ with $\forall w^{i+1}(x^i \mathrel{\varepsilon} w^{i+1} \leftrightarrow y^i \mathrel{\varepsilon} w^{i+1})$, where w^{i+1} is a variable not appearing in σ. Since $T \vdash \forall x^i, y^i(\forall w^{i+1}(x^i \mathrel{\varepsilon} w^{i+1} \leftrightarrow y^i \mathrel{\varepsilon} w^{i+1}) \rightarrow x^i = y^i)$, we have that $T \vdash \sigma \leftrightarrow \tau$. Let $x_1^{i_1}, \dots, x_n^{i_n}$ be all the variables appearing in τ, and assume that $i_1 \geq i_2 \geq \cdots \geq i_n$. By renaming variables if necessary, we may assume that the variables $x_1^{i_1+1}, \dots, x_n^{i_n+1}$ do not appear in τ. Using Strong Ambiguity, we may successively raise the type of each variable $x_1^{i_1}, \dots, x_n^{i_n}$, so

$$T + \mathsf{StAmb} \vdash \tau \leftrightarrow (\dots(\tau^{x_1^{i_1}+})^{x_2^{i_2}+}\dots)^{x_n^{i_n}+}.$$

But, $(\dots(\tau^{x_1^{i_1}+})^{x_2^{i_2}+}\dots)^{x_n^{i_n}+}$ is τ^+, so $T + \mathsf{StAmb} \vdash \tau \leftrightarrow \tau^+$, and therefore since $T \vdash \sigma^+ \leftrightarrow \tau^+$ (for the same reason that $T \vdash \sigma \leftrightarrow \tau$), we have that $T + \mathsf{StAmb} \vdash \sigma \leftrightarrow \sigma^+$. □

Let us now establish the connection between tangled tsaus, Strong Ambiguity, and satisfiability of semistratified sentences in $\mathcal{L}_{\mathsf{NF}}$-structures.

Theorem 2.7 *Let $\mathscr{A} = (A_0, A_1, \dots, \{\varepsilon_{i,j}^{\mathscr{A}}\}_{i<j}\}$ be an $\mathcal{L}_{\mathsf{TTT}}$-structure such that for all $i \in \omega$, $\mathscr{A} \models \forall x^i, y^i(\forall w^{i+1}(x^i \mathrel{\varepsilon} w^{i+1} \leftrightarrow y^i \mathrel{\varepsilon} w^{i+1}) \rightarrow x^i = y^i)$ (note that any model of TTT is such a structure). The following are equivalent:*

(i) there is an $\mathcal{L}_{\mathsf{NF}}$-structure $\mathcal{M}$ such that for all $\mathcal{L}_{\mathsf{TTT}}$-sentences σ,

$$\mathcal{M} \models \sigma^* \iff \mathscr{A} \models \sigma.$$

(ii) $\mathscr{A} \models \mathsf{StAmb}$.
(iii) $\mathscr{A}$ has a tangled tsau.

Proof. We show that (i) $\Longrightarrow$ (ii). Let $\mathcal{M}$ be an $\mathcal{L}_{\mathsf{NF}}$-structure such that for all $\mathcal{L}_{\mathsf{TTT}}$-sentences σ, $\mathcal{M} \models \sigma^*$ iff $\mathscr{A} \models \sigma$. Notice that for any $\mathcal{L}_{\mathsf{TTT}}$-sentence σ and variable x^i, $(\sigma^{x^i+})^*$ is σ^*, so $\mathcal{M} \models (\sigma \leftrightarrow \sigma^{x^i+})^*$, which means that $\mathcal{A} \models \sigma \leftrightarrow \sigma^{x^i+}$. Therefore, $\mathcal{A} \models \mathsf{StAmb}$.

Assume now that (ii) holds, i.e., $\mathscr{A} \models \mathsf{StAmb}$. We know that $\mathscr{A} \models \forall x^0 \exists y^0 \forall w^2(x^0 \mathrel{\varepsilon} w^2 \leftrightarrow y^0 \mathrel{\varepsilon} w^2) \wedge \forall y^0 \exists x^0 \forall w^2(x^0 \mathrel{\varepsilon} w^2 \leftrightarrow y^0 \mathrel{\varepsilon} w^2)$, so by Strong Ambiguity we have that

$$\mathscr{A} \models \forall x^0 \exists y^1 \forall w^2(x^0 \mathrel{\varepsilon} w^2 \leftrightarrow y^1 \mathrel{\varepsilon} w^2) \wedge \forall y^1 \exists x^0 \forall w^2(x^0 \mathrel{\varepsilon} w^2 \leftrightarrow y^1 \mathrel{\varepsilon} w^2). \tag{1}$$

Similarly, for all $i > 0$, we know that

$$\mathscr{A} \models \forall x^i \exists y^i \left(\bigwedge_{j<i} (\forall z_j^j (z_j^j \mathrel{\varepsilon} x^i \leftrightarrow z_j^j \mathrel{\varepsilon} y^i)) \wedge \forall w^{i+2}(x^i \mathrel{\varepsilon} w^{i+2} \leftrightarrow y^i \mathrel{\varepsilon} w^{i+2}) \right)$$
$$\wedge \forall y^i \exists x^i \left(\bigwedge_{j<i} (\forall z_j^j (z_j^j \mathrel{\varepsilon} x^i \leftrightarrow z_j^j \mathrel{\varepsilon} y^i)) \wedge \forall w^{i+2}(x^i \mathrel{\varepsilon} w^{i+2} \leftrightarrow y^i \mathrel{\varepsilon} w^{i+2}) \right),$$

so by Strong Ambiguity again we have that

$$\mathscr{A} \models \forall x^i \exists y^{i+1} \Big(\bigwedge_{j<i} (\forall z^j_j (z^j_j \,\varepsilon\, x^i \leftrightarrow z^j_j \,\varepsilon\, y^{i+1})) \wedge \forall w^{i+2} (x^i \,\varepsilon\, w^{i+2} \leftrightarrow y^{i+1} \,\varepsilon\, w^{i+2}) \Big)$$
$$\wedge \forall y^{i+1} \exists x^i \Big(\bigwedge_{j<i} (\forall z^j_j (z^j_j \,\varepsilon\, x^i \leftrightarrow z^j_j \,\varepsilon\, y^{i+1})) \wedge \forall w^{i+2} (x^i \,\varepsilon\, w^{i+2} \leftrightarrow y^{i+1} \,\varepsilon\, w^{i+2}) \Big). \quad (2)$$

By our assumption about $\mathscr{A}$, we know that for all $i \in \omega$, $\mathscr{A} \models \forall x^i, y^i (\forall w^{i+1} (x^i \,\varepsilon\, w^{i+1} \leftrightarrow y^i \,\varepsilon\, w^{i+1}) \to x^i = y^i)$, so by Strong Ambiguity, we also have that $\mathscr{A} \models \forall x^i, y^i (\forall w^{i+2} (x^i \,\varepsilon\, w^{i+2} \leftrightarrow y^i \,\varepsilon\, w^{i+2}) \to x^i = y^i)$. Therefore, from (1) we get that

$$\mathscr{A} \models \forall x^0 \exists ! y^1 \forall w^2 (x^0 \,\varepsilon\, w^2 \leftrightarrow y^1 \,\varepsilon\, w^2) \wedge \forall y^1 \exists ! x^0 \forall w^2 (x^0 \,\varepsilon\, w^2 \leftrightarrow y^1 \,\varepsilon\, w^2), \quad (3)$$

and from (2) that

$$\mathscr{A} \models \forall x^i \exists ! y^{i+1} \Big(\bigwedge_{j<i} (\forall z^j_j (z^j_j \,\varepsilon\, x^i \leftrightarrow z^j_j \,\varepsilon\, y^{i+1})) \wedge \forall w^{i+2} (x^i \,\varepsilon\, w^{i+2} \leftrightarrow y^{i+1} \,\varepsilon\, w^{i+2}) \Big)$$
$$\wedge \forall y^{i+1} \exists ! x^i \Big(\bigwedge_{j<i} (\forall z^j_j (z^j_j \,\varepsilon\, x^i \leftrightarrow z^j_j \,\varepsilon\, y^{i+1})) \wedge \forall w^{i+2} (x^i \,\varepsilon\, w^{i+2} \leftrightarrow y^{i+1} \,\varepsilon\, w^{i+2}) \Big). \quad (4)$$

Let $f_0 : A_0 \to A_1$ be such that for all $x \in A_0$, $f_0(x)$ is the unique $y \in A_1$ for which $\mathscr{A} \models \forall w^2 (x \,\varepsilon\, w^2 \leftrightarrow y \,\varepsilon\, w^2)$. By definition, we have that for all $x \in A_0$ and $w \in A_2$,

$$x \,\varepsilon^{\mathscr{A}}_{0,2}\, w \iff f_0(x) \,\varepsilon^{\mathscr{A}}_{1,2}\, w. \quad (5)$$

Similarly, for $i > 0$, let $f_i : A_i \to A_{i+1}$ be such that for all $x \in A_i$, $f_i(x)$ is the unique $y \in A_{i+1}$ for which $\mathscr{A} \models \bigwedge_{j<i} (\forall z^j_j (z^j_j \,\varepsilon\, x \leftrightarrow z^j_j \,\varepsilon\, y)) \wedge \forall w^{i+2} (x \,\varepsilon\, w^{i+2} \leftrightarrow y \,\varepsilon\, w^{i+2})$. Again, by definition, we have that for all $j < i < \omega$, $x \in A_i$, and $z \in A_j$,

$$z \,\varepsilon^{\mathscr{A}}_{j,i}\, x \iff z \,\varepsilon^{\mathscr{A}}_{j,i+1}\, f_i(x), \quad (6)$$

and for all $w \in A_{i+2}$,

$$x \,\varepsilon^{\mathscr{A}}_{i,i+2}\, w \iff f_i(x) \,\varepsilon^{\mathscr{A}}_{i+1,i+2}\, w. \quad (7)$$

Let $f = (f_0, f_1, \dots)$. By (3) and (4), we know that for all $i \in \omega$, f_i is a bijection from A_i to A_{i+1}. Furthermore, for all $i < j < \omega$ and $k < l < \omega$, where $i \leq k$ and $j \leq l$, and for all $x \in A_i$ and $y \in A_j$,

$$\begin{aligned}
x \,\varepsilon^{\mathscr{A}}_{i,j}\, y &\iff x \,\varepsilon^{\mathscr{A}}_{i,j-1}\, f^{-1}_{j-1}(y) && \text{(by (6) if } j > i+2) \\
&\ \ \vdots \\
&\iff x \,\varepsilon^{\mathscr{A}}_{i,i+2}\, f^{-1}_{i+2} \circ \cdots \circ f^{-1}_{j-1}(y) && \text{(by (6) if } j > i+2) \\
&\iff f_i(x) \,\varepsilon^{\mathscr{A}}_{i+1,i+2}\, f^{-1}_{i+2} \circ \cdots \circ f^{-1}_{j-1}(y) && \text{(by (5) if } k > i = 0, \text{ or (7) if } k > i > 0) \\
&\iff \begin{cases} f_i(x) \,\varepsilon^{\mathscr{A}}_{i+1,i+3}\, f^{-1}_{i+3} \circ \cdots \circ f^{-1}_{j-1}(y), & \text{if } j > i+2 \text{ and } k > i \\ f_i(x) \,\varepsilon^{\mathscr{A}}_{i+1,i+3}\, f_j(y), & \text{if } l > j = i+2 \text{ and } k > i \end{cases} && \text{(by (6))} \\
&\ \ \vdots && \text{(by alternating (7) and (6), followed by a number of (6) if necessary)} \\
&\iff f_{k-1} \circ \cdots \circ f_i(x) \,\varepsilon^{\mathscr{A}}_{k,l}\, f_{l-1} \circ \cdots \circ f_j(y).
\end{aligned}$$

So, f is a tangled tsau of $\mathscr{A}$, i.e., (iii) holds.

Finally, (iii) $\Longrightarrow$ (i) follows from Lemma 2.4. □

3 Permutation models

Assuming that TTT is consistent, Strong Ambiguity is consistent with Ambiguity, but is it independent of it? In other words, does the consistency of TTT imply the consistency of $\mathsf{TTT} + \neg\mathsf{StAmb} + \mathsf{Amb}$? In NF (cf. [5] or [1]), a common way of establishing relative consistency results is through permutation models. We describe a similar technique in the context of Tangled Type Theory.

Definition 3.1 Let $\mathscr{A} = (A_0, A_1, \dots, \{\varepsilon^{\mathscr{A}}_{i,j}\}_{i<j})$ be an $\mathcal{L}_{\mathsf{TTT}}$-structure. We say that $\pi = \{\pi_{i,j}\}_{i<j}$ is a *permutation of* $\mathscr{A}$ if for all $i < j < \omega$, $\pi_{i,j} : A_j \to A_j$ is a permutation that moves finitely many elements of A_j. We let $\pi(\mathscr{A})$ be the $\mathcal{L}_{\mathsf{TTT}}$-structure $(A_0, A_1, \dots, \{\varepsilon^{\pi(\mathscr{A})}_{i,j}\}_{i<j})$, where for all $i < j < \omega$ and $x, y \in A_j$,

$$x\,\varepsilon^{\pi(\mathscr{A})}_{i,j}\,y \iff x\,\varepsilon^{\mathscr{A}}_{i,j}\,\pi_{i,j}(y).$$

The axioms of TTT are preserved by permutations.

Theorem 3.2 *If* $\mathscr{A} = (A_0, A_1, \dots, \{\varepsilon^{\mathscr{A}}_{i,j}\}_{i<j})$ *is a model of* TTT, *and* $\pi = \{\pi_{i,j}\}_{i<j}$ *is a permutation of* $\mathscr{A}$, *then* $\pi(\mathscr{A})$ *is a model of* TTT.

P r o o f . Notice that if $\varrho = \{\varrho_{i,j}\}_{i<j}$ and $\varrho' = \{\varrho'_{i,j}\}_{i<j}$ are permutations of $\mathscr{A}$, then $\tau = \{\varrho_{i,j} \circ \varrho'_{i,j}\}_{i<j}$ is also a permutation of $\mathscr{A}$, and $\tau(\mathscr{A})$ is $\varrho(\varrho'(\mathscr{A}))$. Furthermore, as we know, every permutation that moves finitely many elements can be expressed as a product of finitely many transpositions. It therefore suffices to prove the theorem under the assumption that there are $k' < l' < \omega$ such that $\pi_{k',l'}$ is a transposition, and for all $i < j < \omega$ for which $(i, j) \neq (k', l')$, $\pi_{i,j}$ is the identity.

Let $s : \omega \to \omega$ be strictly increasing and $i \in \omega$. We first show that $\pi(\mathscr{A}) \models (\mathsf{Ext}^{i+1})^s$, i.e., $\pi(\mathscr{A}) \models \forall x^{s(i+1)}, y^{s(i+1)}(x^{s(i+1)} = y^{s(i+1)} \leftrightarrow \forall z^{s(i)}(z^{s(i)}\,\varepsilon\,x^{s(i+1)} \leftrightarrow z^{s(i)}\,\varepsilon\,y^{s(i+1)}))$. Let $x, y \in A_{s(i+1)}$ such that for all $z \in A_{s(i)}$, $z\,\varepsilon^{\pi(\mathscr{A})}_{s(i),s(i+1)}\,x$ iff $z\,\varepsilon^{\pi(\mathscr{A})}_{s(i),s(i+1)}\,y$, i.e., $z\,\varepsilon^{\mathscr{A}}_{s(i),s(i+1)}\,\pi_{s(i),s(i+1)}(x)$ iff $z\,\varepsilon^{\mathscr{A}}_{s(i),s(i+1)}\,\pi_{s(i),s(i+1)}(y)$. We know that $\mathscr{A} \models (\mathsf{Ext}^{i+1})^s$, so $\pi_{s(i),s(i+1)}(x) = \pi_{s(i),s(i+1)}(y)$, and therefore since $\pi_{s(i),s(i+1)}$ is 1-1, we have that $x = y$.

We now verify that $\pi(\mathscr{A}) \models (\mathsf{Co}^{i+1})^s$. Let $\varphi(x^i, u_1^{i_1}, \dots, u_n^{i_n})$ be an $\mathcal{L}_{\mathsf{TST}}$-formula, where y^{i+1} is not free in φ. Let $u_1 \in A_{s(i_1)}, \dots, u_n \in A_{s(i_n)}$. We show that there exists some $y \in A_{s(i+1)}$ such that for all $x \in A_{s(i)}$,

$$x\,\varepsilon^{\mathscr{A}}_{s(i),s(i+1)}\,\pi_{s(i),s(i+1)}(y) \iff x\,\varepsilon^{\pi(\mathscr{A})}_{s(i),s(i+1)}\,y \iff \pi(\mathscr{A}) \models \varphi^s(x, u_1, \dots, u_n). \tag{8}$$

If $k' \notin \operatorname{ran}(s)$ or $l' \notin \operatorname{ran}(s)$, then by our assumption about π, we have that for every $j_1 < j_2 < \omega$, $\varepsilon^{\pi(\mathscr{A})}_{s(j_1),s(j_2)}$ is $\varepsilon^{\mathscr{A}}_{s(j_1),s(j_2)}$, so $\pi(\mathscr{A}) \models \varphi^s(x, u_1, \dots, u_n)$ is equivalent to $\mathscr{A} \models \varphi^s(x, u_1, \dots, u_n)$, and therefore (8) holds because $\mathscr{A} \models (\mathsf{Co}^{i+1})^s$. Assume now that $k' = s(k)$ and $l' = s(l)$, for some $k < l < \omega$. Let $\psi(x^i, u_1^{i_1}, \dots, u_n^{i_n}, v_1^l, v_2^l)$ be the $\mathcal{L}_{\mathsf{TST}}$-formula we get from φ if for all variables w^k, z^l, we replace each occurrence of $w^k\,\varepsilon\,z^l$ with

$$(z^l = v_1^l \to w^k\,\varepsilon\,v_2^l) \wedge (z^l = v_2^l \to w^k\,\varepsilon\,v_1^l) \wedge (z^l \neq v_1^l \wedge z^l \neq v_2^l \to w^k\,\varepsilon\,z^l).$$

Let v_1, v_2 be the two elements of A_l that are moved by $\pi_{s(k),s(l)}$. It is easy to see that for all $x \in A_{s(i)}$,

$$\pi(\mathscr{A}) \models \varphi^s(x, u_1, \dots, u_n) \iff \mathscr{A} \models \psi^s(x, u_1, \dots, u_n, v_1, v_2). \tag{9}$$

Since $\mathscr{A} \models (\mathsf{Co}^{i+1})^s$, we know that there exists some $y \in A_{s(i+1)}$ such that for all $x \in A_{s(i)}$,

$$x\,\varepsilon^{\mathscr{A}}_{s(i),s(i+1)}\,y \iff \mathscr{A} \models \psi^s(x, u_1, \dots, u_n, v_1, v_2).$$

Therefore, by (9) and the fact that $\pi_{s(i),s(i+1)}$ is onto, we get that there exists some $y \in A_{s(i+1)}$ such that for all $x \in A_{s(i)}$, (8) holds. Hence, $\pi(\mathscr{A}) \models (\forall u_1^{i_1}, \dots, u_n^{i_n} \exists y^{i+1} \forall x^i (x^i\,\varepsilon\,y^{i+1} \leftrightarrow \varphi(x^i, u_1^{i_1}, \dots, u_n^{i_n})))^s$. □

A permutation $\pi = \{\pi_{i,j}\}_{i<j}$ of an $\mathcal{L}_{\mathsf{TTT}}$-structure $\mathscr{A} = (A_0, A_1, \dots, \{\varepsilon^{\mathscr{A}}_{i,j}\}_{i<j})$ acts on it by permuting the extensions of certain sets. Note, however, that for $i > 0$, every set $x \in A_i$ has an extension with respect to each A_j for $j < i$ (let us call it j-extension). Moreover, Strong Ambiguity implies that all these extensions of x must satisfy the same definable properties. Therefore, if for some $j < k < \omega$, π rearranges the j-extension and the k-extension of x so that they differ in some definable way (for example, one is empty and the other one is non-empty), then $\pi(\mathscr{A})$ cannot satisfy Strong Ambiguity. Below, we use this argument to show that Strong Ambiguity is independent of $\mathsf{TTT} + \mathsf{Amb}$.

Theorem 3.3 *If* TTT *is consistent, then*

(i) TTT + StAmb *is consistent.*

(ii) TTT + ¬StAmb + Amb *is consistent.*

Proof. We prove (i). Since TTT is consistent, we know that there is an NF-model $\mathcal{M} = (M, \varepsilon^{\mathcal{M}})$. Let $\mathscr{A} = (M, M, \dots, \{\varepsilon^{\mathscr{A}}_{i,j}\}_{i<j}\}$ be the $\mathcal{L}_{\mathsf{TTT}}$-structure where each $\varepsilon^{\mathscr{A}}_{i,j}$ is $\varepsilon^{\mathcal{M}}$. It is easy to verify that $\mathscr{A}$ is a model of TTT. Moreover, $\mathscr{A}$ satisfies StAmb because the sequence $f = (f_0, f_1, \dots)$, where each f_i is the identity on M, is clearly a tangled tsau of $\mathscr{A}$.

We now show that (ii) holds. Let $\mathscr{A}$ be as above. We know that $\mathscr{A}$ is a model of TTT + StAmb, and therefore of TTT + Amb. Let $\pi_{0,2} : A_2 \to A_2$ be the permutation that permutes the elements $w_0, w_1 \in A_2$, where $\mathscr{A} \models \forall u^1(u^1 \not\varepsilon\, w_0)$ and $\mathscr{A} \models \forall u^1(u^1\, \varepsilon\, w_1)$. Notice that $\mathscr{A} \models \forall u^0(u^0 \not\varepsilon\, w_0)$ since $\varepsilon^{\mathscr{A}}_{0,2}$ is $\varepsilon^{\mathscr{A}}_{1,2}$ (they are both $\varepsilon^{\mathcal{M}}$). For all $i < j < \omega$, where $(i, j) \neq (0, 2)$, let $\pi_{i,j} : A_j \to A_j$ be the identity. Obviously, $\pi = \{\pi_{i,j}\}_{i<j}$ is a permutation of $\mathscr{A}$, so $\pi(\mathscr{A})$ is a model of TTT. Furthermore, $\pi(\mathscr{A})$ satisfies the same $\mathcal{L}_{\mathsf{TST}}$-sentences as $\mathscr{A}$ because for every $i \in \omega$, $\varepsilon^{\pi(\mathscr{A})}_{i,i+1}$ is $\varepsilon^{\mathscr{A}}_{i,i+1}$. Therefore, $\pi(\mathscr{A})$ satisfies Amb. Finally, suppose that $\pi(\mathscr{A}) \models$ StAmb. We know that $\pi(\mathscr{A}) \models \forall z^2(\exists y^0(y^0\, \varepsilon\, z^2) \leftrightarrow \exists x^0(x^0\, \varepsilon\, z^2))$, so by Strong Ambiguity, we have that $\pi(\mathscr{A}) \models \forall z^2(\exists y^1(y^1\, \varepsilon\, z^2) \leftrightarrow \exists x^0(x^0\, \varepsilon\, z^2))$. But, $\pi(\mathscr{A}) \models \exists y^1(y^1\, \varepsilon\, w_1)$ because $\exists y \in A_1(y\, \varepsilon^{\mathscr{A}}_{1,2}\, w_1)$, whereas $\pi(\mathscr{A}) \not\models \exists x^0(x^0\, \varepsilon\, w_1)$ because $\nexists x \in A_0(x\, \varepsilon^{\mathscr{A}}_{0,2}\, \pi_{0,2}(w_1))$. Hence, $\pi(\mathscr{A}) \not\models$ StAmb. □

The argument we just used does not work for the following question.

Question 1 Assume that TTT is consistent. Is TTT + ¬Amb consistent?

Notice that if TTT + ¬Amb is inconsistent, then NF and TTT must have the same stratified theorems (assuming we ignore type indices), i.e., for all $\mathcal{L}_{\mathsf{TST}}$-sentences σ,

$$\mathsf{NF} \vdash \sigma^* \iff \mathsf{TTT} \vdash \sigma. \tag{10}$$

Let us prove it. The right to left implication is obvious because as we noted above, every model $\mathcal{M}$ of NF can be turned into a TTT-model $\mathscr{A}$ such that for all $\mathcal{L}_{\mathsf{TST}}$-sentences σ, $\mathcal{M} \models \sigma^*$ iff $\mathscr{A} \models \sigma$. Now, for the left to right implication, let σ be an $\mathcal{L}_{\mathsf{TST}}$-sentence, and assume that $\mathsf{NF} \vdash \sigma^*$. Let $\mathscr{A} = (A_0, A_1, \dots, \{\varepsilon^{\mathscr{A}}_{i,j}\}_{i<j})$ be a model of TTT. We know that $\mathscr{A} \models$ Amb, so $\mathscr{B} = (A_0, A_1, \dots, \{\varepsilon^{\mathscr{A}}_{i,i+1}\}_{i\in\omega})$ is a model of TST + Amb, and therefore by Specker's theorem, there is some elementarily equivalent model $\mathscr{C} = (C_0, C_1, \dots, \{\varepsilon^{\mathscr{C}}_{i,i+1}\}_{i\in\omega})$ of TST with a tsau f. Let $\mathcal{M} = (C_0, \varepsilon^{\mathcal{M}})$, where for all $x, y \in C_0$, $x\, \varepsilon^{\mathcal{M}}\, y$ iff $x\, \varepsilon^{\mathscr{C}}_{0,1}\, f_0(y)$. It is easy to show that for all $\mathcal{L}_{\mathsf{TST}}$-sentences τ, $\mathcal{M} \models \tau^*$ iff $\mathscr{C} \models \tau$. So, $\mathscr{A} \models \sigma$.

In his proof for the equiconsistency of TTT and NF (cf. [2]), Holmes showed that for any model $\mathscr{A} = (A_0, A_1, \dots, \{\varepsilon^{\mathscr{A}}_{i,j}\}_{i<j})$ of TTT, there is a strictly increasing $r : \omega \to \omega$ such that the $\mathcal{L}_{\mathsf{TTT}}$-structure $\mathscr{A}_r = (A_{r(0)}, A_{r(1)}, \dots, \{\varepsilon^{\mathscr{A}}_{r(i),r(j)}\}_{i<j})$ is a model of TTT + MStAmb, where MStAmb is a slightly stronger version of Ambiguity (we call it Mildly Strong Ambiguity) defined as the set of all $\sigma \leftrightarrow \sigma^s$, where $s : \omega \to \omega$ is strictly increasing and σ is an $\mathcal{L}_{\mathsf{TST}}$-sentence. Obviously, Mildly Strong Ambiguity implies Ambiguity and is implied by Strong Ambiguity, but what about the reverse implications?

Question 2 Assume that TTT is consistent. Is TTT + ¬StAmb + MStAmb consistent? Is TTT + ¬MStAmb + Amb consistent?

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