Computational Aspects of Equivariant Hilbert Series of Canonical Rings for Algebraic Curves

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Abstract. We study computational aspects of the problem of decomposing finite group actions on graded modules arising in arithmetic geometry, in the context of ordinary representation theory. We provide an algorithm to compute the equivariant Hilbert series of automorphisms acting on canonical rings of projective curves, using the formulas of Chevalley and Weil. Further, we apply our results on Fermat curves, determine explicitly the respective equivariant Hilbert series and extend the computation to the short exact sequence that arises from Petri's Theorem. Finally, we implement the above computations in Sage.

Keywords: Hilbert series · Group actions · Holomorphic differentials · Fermat curves

1 Introduction

1.1 Equivariant Hilbert Series

One of the most fundamental problems in representation theory of finite groups is that of decomposing representations into direct sums of indecomposables. Namely, given a finite group G acting on a vector space V over an arbitrary field k, the problem amounts to determining, for each indecomposable representation $W \in \mathrm{Ind}(G)$ over k, a natural number $n_{W,V}$ such that $V = \bigoplus_{W\in \mathrm{Ind}(G)} n_{W,V} W$. In the context of modular representation theory, that is, if

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the characteristic of the ground field is positive and divides the order of G, there are several complications that make the general case of this problem practically impossible; however, if $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p \nmid |G|$, every indecomposable representation is irreducible, and there is a direct approach using character theory

$$n_{W,V} = \langle \chi_V, \chi_W \rangle := \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_W(g)},$$

where χ_V denotes the character of the representation $\rho : G \to \mathrm{GL}(V)$.

The above technique can be also used when one generalizes the objects acted on, from vector spaces V over k to modules M over some k-algebra R. Historically, a case of particular interest is that of finite groups acting as automorphisms on polynomial rings: the study of their G-structure is essentially the main motivation behind the development of invariant theory, a subject whose origins date back to Hilbert's fourteenth problem. The next level of abstraction dictates to consider, instead of a polynomial ring, an arbitrary graded, Noetherian k-algebra $R = \bigoplus_{d=0}^{\infty} R_d$ acted upon by a finite group G. Since each graded piece R_d is a vector space over k, one can apply the techniques of the first paragraph to obtain for each $d \in \mathbb{N}$ and each $W \in \mathrm{Irr}(G)$, natural numbers $n_{W,d}$ such that $R_d = \bigoplus_{W \in \mathrm{Irr}(G)} n_{W,d} W$. The decomposition of R is then given by

$$R = \bigoplus_{d=0}^{\infty} R_d = \bigoplus_{d=0}^{\infty} \bigoplus_{W \in \mathrm{Irr}(G)} n_{W,d} W = \bigoplus_{W \in \mathrm{Irr}(G)} \sum_{d=0}^{\infty} n_{W,d} W.$$

One obtains for each $W \in \mathrm{Irr}(G)$ a generating function for the sequence $\{n_{W,d}\}_{d=0}^{\infty}$

$$H_{R,W}(T) = \sum_{d=0}^{\infty} n_{W,d} T^d.$$

By studying the convergence of $H_{R,W}(T)$, the infinite information of the action of G on R, which is infinite dimensional over k, can be packaged in a finite sequence $\{H_{R,W}(T) \mid W \in \mathrm{Irr}(G)\}$ which is called *the equivariant Hilbert series* of the pair (R, G). The best understood case is, again, that of polynomial rings: if $R = \mathrm{Sym}\,(V)$ is the symmetric algebra of a finite dimensional k-vector space V, Molien's theorem [15, Theorem 2.1] says that

$$H_{R,W}(T) = \frac{\dim W}{|G|} \sum_{g \in G} \frac{\overline{\chi_W(g)}}{\det\left(\mathrm{Id}_V - gT\right)}. \tag{1}$$

Of course, hoping to obtain an analogous formula for arbitrary graded, Noetherian k-algebras R, is unrealistic, unless one has some concrete information on the action of G on R. Since graded, Noetherian k-algebras arise as homogeneous coordinate rings of projective varieties, this can be achieved by switching the viewpoint towards algebraic geometry.

1.2 Petri's Theorem

From now on we assume that k is algebraically closed. Let X be a smooth, projective curve of genus g over k. Recall that X does not come a priori with a fixed embedding into projective space; however, it is well known that explicit projective embeddings can be constructed using (very ample) line bundles on X. Of all possible projective embeddings of X, there is one that stands out as *canonical*: that determined by the cotangent bundle Ω_X, referred to also as the sheaf of holomorphic differentials on X. It is given by

$$X \to \mathbb{P}\left(H^0(X, \Omega_X)\right) \cong \mathbb{P}_k^{g-1}, \quad P \mapsto [\omega_1(P) : \cdots : \omega_g(P)],$$

where $\{\omega_1, \ldots, \omega_g\}$ denotes a k-basis for the global sections $H^0(X, \Omega_X)$.

To see that this construction gives an embedding, we rephrase the above in the algebraic language. Recall that the homogeneous coordinate ring of the projectivization $\mathbb{P}\left(H^0(X, \Omega_X)\right)$ is the symmetric algebra $\mathrm{Sym}\left(H^0(X, \Omega_X)\right)$, which may be identified with a polynomial ring in g variables. The canonical embedding is then determined by the so-called canonical map, as ensured by the following classic theorem [14] due to Max Noether, Federigo Enriques and Karl Petri.

Theorem 1. *If X is not hyperelliptic and has genus $g \geq 4$, the canonical map*

$$\phi : S := \mathrm{Sym}\left(H^0(X, \Omega_X)\right) \to S_X := \bigoplus_{m=0}^{\infty} H^0(X, \Omega_X^{\otimes m}),$$

is surjective. Its kernel I_X, the canonical ideal, is generated in degrees 2 *and* 3.

Quoting from [3, Section 2, §3], the *canonical ring* S_X "is the homogeneous coordinate ring of the canonically embedded curve X". Any action of a finite group G on X induces an action on S_X, and thus, we may seek a formula for its equivariant Hilbert series. Assuming that $\mathrm{char}(k) = p \nmid |G|$, we may use Molien's formula to compute the respective series for S and thus obtain the equivariant Hilbert series for the canonical ideal I_X. It is worth noting that these calculations are the starting point in computing the action of G on the minimal graded resolution of S_X as an S-module. The latter is well-studied in the non-equivariant case mainly due its connection to Green's syzygy conjecture [5]; we hope that this work will shed some light to possible generalizations in the equivariant case.

The main results of this paper are:

1. General formulas (Theorem 3) and an algorithm (Algorithm 2) that gives the equivariant Hilbert series of S_X for arbitrary curves X.
2. Explicit formulas (Theorem 4) for the equivariant Hilbert series of S_X when X is a Fermat curve.
3. A Sage [16] program[1,2] that computes, when X is a Fermat curve:
   (a) $\{H_{S_X,V}(T) : V \in \mathrm{Irr}(G)\}$, by implementing the formulas of Theorem 4.
   (b) $\{H_{S,V}(T) : V \in \mathrm{Irr}(G)\}$, by implementing Molien's formula.

[1] http://users.uoa.gr/ kontogar/Code/EquivariantSage.ipynb.
[2] http://users.uoa.gr/ kontogar/Code/EquivariantSage.pdf.

(c) $\{H_{I_X,V}(T) : V \in \mathrm{Irr}(G)\}$, subtracting the two above results.

We remark that similar results were obtained in our preprint [1] using different techniques. We have verified computationally that the two approaches lead to the same answers; a concrete theoretical proof involves complicated calculations, however we can indicatively provide the reader with one, i.e., for one of the irreducible representations, upon request.

2 Equivariant Hilbert Series of Canonical Rings

Let X be a smooth, projective curve of genus g over an algebraically closed field k of arbitrary characteristic $p \geq 0$. Let G be a finite subgroup of its automorphism group $\mathrm{Aut}_k(X)$ of order $|G|$ not divisible by p. For $m \geq 1$, we denote by $\Omega_X^{\otimes m}$ the sheaf of holomorphic m-differentials on X and by W_m the k-vector space $H^0(X, \Omega_X^{\otimes m})$ of its global sections. By the Riemann-Roch Theorem [6, IV.1.3],

$$\dim_k W_m = \begin{cases} 1 & , \text{if } m = 0 \\ g & , \text{if } m = 1 \\ (2m-1)(g-1) & , \text{if } m \geq 2 \end{cases}$$

it is further well known that the action of G on X induces an action on W_m for all $m \geq 1$. Let $\mathrm{Irr}(G)$ denote the group of irreducible representations of G over k; the isomorphism class of each W_m, viewed as a kG-module, is uniquely determined by a collection of integers $\{N_{V,m}\}_{V \in \mathrm{Irr}(G)}$ such that $W_m = \bigoplus_{V \in \mathrm{Irr}(G)} N_{V,m} V$. The classic approach to computing the integers $N_{V,m}$ goes as follows.

Algorithm 1: Computing the multiplicities $N_{V,m}$.

Inputs:

1. The character table $[\chi_V(g)]_{\substack{V \in \mathrm{Irr}(G) \\ g \in G}}$ of G over k.
2. The action of G on the closed points of X.
3. A k-basis $\mathbf{b}_m = \{f(x,y)dx^{\otimes m}\}$ for W_m.

Output: A list of integers $\{N_{V,m}\}_{V \in \mathrm{Irr}(G)}$ such that $W_m = \bigoplus_{V \in \mathrm{Irr}(G)} N_{V,m} V$.
Method:

1. For each $g \in G$
   (a) Compute the matrix $\rho(g)$, given by the action of g on the basis $\mathbf{b}_m$.
   (b) Produce a list $\{\chi_{W_m}(g) : g \in G\}$ where $\chi_{W_m}(g) = \mathrm{Trace}\,(\rho(g))$.
2. For each $V \in \mathrm{Irr}(G)$, compute

$$N_{V,m} = \frac{1}{|G|} \sum_{g \in G} \chi_{W_m}(g) \overline{\chi_V(g)}.$$

The downside of the above algorithm comes from input (3), in that there does not exist a general method to compute explicit bases for the k-vector spaces

$W_m = H^0(X, \Omega_X^{\otimes m})$. Even in the few cases in which bases are known, one of which is that of Fermat curves that we will study in Sect. 3, the sums in step (2) can in practice become rather difficult to compute, see for example our proof of [1, Theorem 20]. An alternative approach, exploited with great success by many authors, see for example [2,4] and [8], is to express the multiplicities $N_{V,m}$ in terms of the ramification data of the action of G on X. The resulting formulas are much easier to use, both in terms of the input required and in terms of computational complexity; however, as is usually the case in such situations, they require some familiarity with technical aspects of arithmetic geometry, which we briefly recall here. For more details the reader may refer to [6, Chapter IV], [12, Chapters 4 & 10], or [13].

From now on, we assume that the characteristic of k is either 0 or does not divide $|G|$. Let $Y = X/G$ be the quotient of X by the action of G. The quotient map $\pi : X \to Y$ is a non-constant, regular morphism of curves of degree $|G|$, so that the number of points in a generic fiber $\pi^{-1}(Q), Q \in Y$ is equal to $|G|$. There exists a finite set of points $Q \in Y$ for which the fiber $\pi^{-1}(Q)$ has cardinality strictly less than $|G|$, called the *branch locus* of π and denoted by $\mathscr{B}$. The ramification locus of π is $\mathscr{R} = \pi^{-1}(\mathscr{B}) \subseteq X$. By [7, Theorem 11.49], the decomposition group of a point $P \in X$ is the cyclic group $G_P = \{\sigma \in G : \sigma(P) = P\}$ and its order is called the *ramification index* of $P \in X$. Since the ramification index is the same for all points in the orbit of $P \in X$, we denote it by e_Q, where $Q = \pi(P) \in Y$. The cyclic group G_P has e_Q-many, distinct, one-dimensional irreducible representations, determined by their characters. Fix ζ_{e_Q} to be a primitive e_Q-th root of unity; the irreducible characters of G_P are all of the of the form χ_P^d, $1 \leq d \leq e_Q$, where χ_P, is the fundamental character at the point P, that is, the character obtained by letting G_P act on a local uniformizer u_P at P considered modulo u_P^2. The monodromy element σ_P is a generator of G_P such that $\sigma_P(u_P) = \zeta_{e_Q} u_P$. For each irreducible representation V of G, we denote by $n_{d,Q,V}$ the multiplicity of the irreducible character χ_P^d in the decomposition of the restricted representation $\mathrm{Res}_{G_P}^G(V)$, i.e., $n_{d,Q,V} = \langle \chi_P^d, \mathrm{Res}_{G_P}^G(\chi_V)\rangle$. We summarize the above in the table below.

Table 1. Notation for the ramification data of the action of G on X

$Y = X/G$	Quotient of X by the action of G
$\mathscr{R}$	Ramification locus of $\pi : X \to Y$
$\mathscr{B}$	Branch locus $\pi : X \to Y$
e_Q	Ramification index at $Q \in \mathscr{B}$
$G_P = \{\sigma \in G : \sigma(P) = P\}$	decomposition group at $P \in \mathscr{R}$
σ_P	monodromy generator of $G_P,\ P \in \mathscr{R}$
$\{\chi_P^d\ :\ 0 \leq d \leq e_Q - 1\}$	irreducible characters of $G_P \cong \mathbb{Z}/e_Q\mathbb{Z}$
$\{n_{d,Q,V}\ : Q \in \mathscr{B},\ 0 \leq d \leq e_Q - 1,\ V \in \mathrm{Irr}(G)\}$	multiplicities of χ_P^d in $\mathrm{Res}_{G_P}^G(V)$

The following result gives an explicit formula for the multiplicities $N_{V,m}$.

Theorem 2 (Chevalley-Weil [2]). *For each $V \in \mathrm{Irr}(G)$, we have that*

$$N_{V,m} = E_{V,m} + (2m-1)(g_Y - 1)\dim V + \sum_{Q \in \mathscr{B}} \sum_{d=0}^{e_Q - 1} \left((m-1)\left(1 - \frac{1}{e_Q}\right) + \left\langle \frac{m-1-d}{e_Q} \right\rangle \right) n_{d,Q,V},$$

where $\mathscr{B}, e_Q$ and $n_{d,Q,V}$ are given in Table 1, g_Y is the genus of Y,

$$E_{V,m} = \begin{cases} N_{V^*,1} & , \text{ if } m = 0 \ (V^* \text{ denotes the dual of } V) \\ 1 & , \text{ if } m = 1 \text{ and } V \text{ is the trivial representation} \\ 0 & , \text{ otherwise.} \end{cases}$$

and $\langle x \rangle = x - \lfloor x \rfloor$ denotes the fractional part of x.

Remark 1. For a proof of the above, see [4, Th. 3.8 & Rem. 3.9] The authors compute the multiplicity of V in the equivariant Euler characteristic $[H^0(X, \Omega_X^{\otimes m})] - [H^1(X, \Omega_X^{\otimes m})]$. The formula for $E_{V,m}$, which is the multiplicity in $H^1(X, \Omega_X^{\otimes m})$, follows from the Riemann-Roch theorem combined with Serre's duality. It is worth mentioning that the above result was generalized in [8] to the weakly ramified case.

Theorem 3. *The equivariant Hilbert series of $S_X = \bigoplus_m H^0(X, \Omega_X^{\otimes m})$*

$$H_{S_X,V}(T) = \sum_{m=0}^{\infty} N_{V,m} T^m$$

of an irreducible representation V of G is given by the rational function

$$H_{S_X,V}(T) = N_{V^*,1} + \delta_V T + \frac{3T-1}{(1-T)^2}(g_Y - 1)\dim V + \frac{T}{(1-T)^2}\dim V |\mathscr{B}| - \frac{1}{1-T}\sum_Q \frac{f'_{Q,V}(1)}{e_Q} - \frac{T}{1-T}\sum_Q \frac{f_{Q,V}(T)}{1-T^{e_Q}},$$

where $\delta_V = 1$ for $V = V_{\mathrm{triv}}$ and 0 otherwise,

$$f_{Q,V}(T) = \sum_{d=0}^{e_Q-1} n_{d,Q,V} T^d$$

and $|\mathscr{B}|$ denotes the cardinality of the branch locus of the cover $X \to X/G$.

Our computations will be in two steps. Write

$$H_{S_X,V}(T) = N_{V^*,1} + \delta_V T + F_V(T) + G_V(T),$$

where

$$F_V(T) = \sum_{m=0}^{\infty} \left((2m-1)(g_Y - 1)\dim V + \sum_Q \sum_{d=0}^{e_Q-1} (m-1)\left(1 - \frac{1}{e_Q}\right) n_{d,Q,V} \right) T^m$$

$$G_V(T) = \sum_{m=0}^{\infty} \sum_Q \sum_{d=0}^{e_Q-1} n_{d,Q,V} \left\langle \frac{m-d-1}{e_Q} \right\rangle T^m. \tag{2}$$

Lemma 1.

$$F_V(T) = \frac{3T-1}{(1-T)^2}(g_Y - 1)\dim V + \frac{2T-1}{(1-T)^2}\dim V \sum_Q \left(1 - \frac{1}{e_Q}\right).$$

Proof. The result follows from the well-known formulas

$$\sum_{m=0}^{\infty}(2m-1)T^m = \frac{3T-1}{(1-T)^2}, \quad \text{and} \quad \sum_{m=0}^{\infty}(m-1)T^m = \frac{2T-1}{(1-T)^2},$$

as well as the fact that $\sum_{d=0}^{e_Q-1} n_{d,Q,V} = \dim V$.

To compute $G_V(T)$, we first prove the following auxiliary lemma.

Lemma 2. *For $A \in \mathbb{Z}$ and $1 < e \in \mathbb{N}$, we have that*

$$\sum_{m=0}^{\infty} \left\langle \frac{m+A}{e} \right\rangle T^m = \frac{T}{e(1-T)^2} + \frac{v_A}{e(1-T)} - \frac{T^{e-v_A}}{(1-T^e)(1-T)},$$

where v_A is the remainder of the division of A by e.

Proof. Recall that $\langle x \rangle = x - \lfloor x \rfloor$. Write $m = \pi e + v$ and $A = \pi_A e + v_A$, for $0 \le v, v_A < e$ and $\pi, \pi_A \in \mathbb{Z}$. Then

$$\left\langle \frac{m+A}{e} \right\rangle = \frac{v+v_A}{e} - \left\lfloor \frac{v+v_A}{e} \right\rfloor,$$

and thus

$$\sum_{m=0}^{\infty} \left\langle \frac{m+A}{e} \right\rangle T^m = \sum_{v=0}^{e-1}\sum_{\pi=0}^{\infty} \left(\frac{v+v_A}{e} - \left\lfloor \frac{v+v_A}{e} \right\rfloor \right) (T^e)^{\pi}\, T^v$$
$$= \frac{1}{1-T^e} \sum_{v=0}^{e-1} \left(\frac{v+v_A}{e} - \left\lfloor \frac{v+v_A}{e} \right\rfloor \right) T^v.$$

Next, we remark that since

$$\left\lfloor \frac{v+v_A}{e} \right\rfloor = \begin{cases} 0 & \text{if } 0 \le v + v_A \le e-1 \\ 1 & \text{if } e \le v + v_A < 2e \end{cases},$$

we have that

$$\sum_{v=0}^{e-1}\left(\frac{v+v_A}{e}-\left\lfloor\frac{v+v_A}{e}\right\rfloor\right)T^v=\sum_{v=0}^{e-1}\frac{v+v_A}{e}T^v-\sum_{v=e-v_A}^{e-1}T^v$$
$$=\frac{1}{e}\sum_{v=0}^{e-1}vT^v+\frac{v_A}{e}\sum_{v=0}^{e-1}T^v-T^{e-v_A}\sum_{v=0}^{v_A-1}T^v.$$

Each of the three sums is given by

$$\frac{1}{e}\sum_{v=0}^{e-1}vT^v=\frac{eT^{e+1}-T^{e+1}-eT^e+T}{e(1-T)^2}=-\frac{T^e}{(1-T)}+\frac{T(1-T^e)}{e(1-T)^2},$$
$$\frac{v_A}{e}\sum_{v=0}^{e-1}T^v=\frac{v_A(1-T^e)}{e(1-T)},$$
$$T^{e-v_A}\sum_{v=0}^{v_A-1}T^v=T^{e-v_A}\frac{1-T^{v_A}}{1-T}=\frac{T^{e-v_A}}{1-T}-\frac{T^e}{1-T}.$$

Observe that the first term of the first sum cancels out with the second term of the third sum, and thus

$$\sum_{m=0}^{\infty}\left\langle\frac{m+A}{e}\right\rangle T^m=\frac{1}{1-T^e}\left(\frac{T(1-T^e)}{e(1-T)^2}+\frac{v_A(1-T^e)}{e(1-T)}-\frac{T^{e-v_A}}{1-T}\right)$$
$$=\frac{T}{e(1-T)^2}+\frac{v_A}{e(1-T)}-\frac{T^{e-v_A}}{(1-T^e)(1-T)}.$$

Corollary 1. *Let* $G_V(T)$ *be as in Eq. (2) and* $f_{Q,V}(T)=\sum_{d=0}^{e_Q-1}n_{d,Q,V}T^d$. *Then*

$$G_V(T)=\frac{\dim V\cdot T}{(1-T)^2}\sum_Q\frac{1}{e_Q}+\frac{\dim V}{1-T}\sum_Q\left(1-\frac{1}{e_Q}\right)$$
$$-\frac{1}{1-T}\sum_Q\frac{f'_{Q,V}(1)}{e_Q}-\frac{T}{1-T}\sum_Q\left(\frac{f_{Q,V}(T)}{1-T^{e_Q}}\right).$$

Proof. Observe that if $0 \le d \le e_Q - 1$, the remainder of the division of $A = -d-1$ by e_Q is $v_A = e_Q - d - 1$. Thus, applying Lemma 2 for $A = -d-1$ we obtain

$$\sum_{m=0}^{\infty}\sum_{Q}\sum_{d=0}^{e_Q-1} n_{d,Q,V}\left\langle \frac{m-d-1}{e_Q}\right\rangle T^m = \sum_{Q}\sum_{d=0}^{e_Q-1} n_{d,Q,V}\sum_{m=0}^{\infty}\left\langle \frac{m-d-1}{e_Q}\right\rangle T^m$$

$$= \sum_{Q}\sum_{d=0}^{e_Q-1} n_{d,Q,V}\left(\frac{T}{e_Q(1-T)^2} + \frac{e_Q-d-1}{e_Q(1-T)} - \frac{T^{d+1}}{(1-T^{e_Q})(1-T)}\right)$$

$$= \sum_{Q}\sum_{d=0}^{e_Q-1} \frac{n_{d,Q,V}T}{e_Q(1-T)^2} + \left(1-\frac{1}{e_Q}\right)\frac{n_{d,Q,V}}{1-T} - \frac{n_{d,Q,V}d}{e_Q(1-T)} - \frac{n_{d,Q,V}T^{d+1}}{(1-T^{e_Q})(1-T)}$$

$$= \sum_{Q} \frac{f_{Q,V}(1)T}{e_Q(1-T)^2} + \left(1-\frac{1}{e_Q}\right)\frac{f_{Q,V}(1)}{1-T} - \frac{f'_{Q,V}(1)}{e_Q(1-T)} - \frac{f_{Q,V}(T)T}{(1-T^{e_Q})(1-T)}.$$

Using again the fact that $f_{Q,V}(1) = \sum_{d=0}^{e_Q-1} n_{d,Q,V} = \dim V$ gives the desired result.

Proof (of Theorem 3). Let $F_V(T)$ and $G_V(T)$ be as in Eq. (2). By Lemma 1 and Corollary 1 we have that

$$F_V(T) = \frac{3T-1}{(1-T)^2}(g_Y-1)\dim V + \frac{2T-1}{(1-T)^2}\dim V\sum_Q\left(1-\frac{1}{e_Q}\right),$$

$$G_V(T) = \frac{\dim V\cdot T}{(1-T)^2}\sum_Q\frac{1}{e_Q} + \frac{\dim V}{1-T}\sum_Q\left(1-\frac{1}{e_Q}\right)$$

$$-\frac{1}{1-T}\sum_Q\frac{f'_{Q,V}(1)}{e_Q} - \frac{T}{1-T}\sum_Q\left(\frac{f_{Q,V}(T)}{1-T^{e_Q}}\right).$$

Adding the second term of $F_V(T)$ to the second term of $G_V(T)$ gives

$$\frac{(2T-1)\dim V}{(1-T)^2}\sum_Q\left(1-\frac{1}{e_Q}\right) + \frac{\dim V}{1-T}\sum_Q\left(1-\frac{1}{e_Q}\right)$$

$$= \frac{\dim V\cdot T}{(1-T)^2}\sum_Q\left(1-\frac{1}{e_Q}\right) = \frac{\dim V\cdot T}{(1-T)^2}\#\mathscr{B} - \frac{\dim V\cdot T}{(1-T)^2}\sum_Q\frac{1}{e_Q},$$

and the last term above cancels out with the first term of $G_V(T)$. Thus

$$F_V(T)+G_V(T) = \frac{3T-1}{(1-T)^2}(g_Y-1)\dim V + \frac{T}{(1-T)^2}\dim V|\mathscr{B}|$$

$$-\frac{1}{1-T}\sum_Q\frac{f'_{Q,V}(1)}{e_Q} - \frac{T}{1-T}\sum_Q\frac{f_{Q,V}(T)}{1-T^{e_Q}}.$$

As a corollary we obtain the below algorithm.

Algorithm 2: Computing the equivariant Hilbert series $\{H_{S_X,V}(T) : V \in \mathrm{Irr}(G)\}$.

Inputs:
1. The character table $[\chi_V(g)]_{\substack{V\in\mathrm{Irr}(G)\\ g\in G}}$ of G over k.
2. The action of G on the closed points of X.

Output: A list of rational functions $\{H_V(t) : V \in \mathrm{Irr}(G)\}$
Method:

1. Compute the ramification locus $\mathscr{R}$ and the branch locus $\mathscr{B}$ of $\pi : X \to Y$.
2. Compute g_Y using the Riemann-Hurwitz formula [17, Theorem 3.4.13].
3. For each $Q \in \mathscr{B}$
   (a) Compute the ramification index e_Q.
   (b) For each $V \in \mathrm{Irr}(G)$ and each $0 \le d \le e_Q - 1$ compute

$$n_{d,Q,V} = \langle \chi_P^d, \mathrm{Res}_{G_P}^G(\chi_V)\rangle \text{ and } f_{Q,V}(T) = \sum_d n_{d,Q,V} T^d.$$

4. For each $V \in \mathrm{Irr}(G)$, compute $H_{S_X,V}(T)$ using Theorem 3.

There are two advantages of Algorithm 2 over Algorithm 1. Firstly it can be used in the cases in which explicit k-bases for polydifferentials are not known; secondly the inner products of step 3(b) are taken over the decomposition groups G_P which are strictly smaller than the full automorphism group G. On the other hand, its disadvantages are that one needs to compute the ramification data of the cover $\pi : X \to Y$, a problem which is wide open in its full generality, and that computing the multiplicities $n_{d,Q,V}$ is not always a straightforward task. We shall demonstrate how this is done in the next section by applying our results to Fermat curves.

3 The Case of Fermat Curves

Let F_n be a Fermat curve with affine model $x^n + y^n + 1 = 0$, defined over an algebraically closed field k of characteristic $p \ge 0$. We assume that $n \ge 4$, $p > 3$ and $n-1$ is not a power of p. To describe the automorphism group $G = \mathrm{Aut}_k(X)$, we write

$$A := \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} = \{\sigma_{\alpha,\beta} : 0 \le \alpha, \beta \le n-1\}$$
$$S_3 = \langle s, t : s^3 = t^2 = 1,\ tst = s^{-1}\rangle = \{1, s, s^2, t, st, ts\}.$$

and note that S_3 acts on A by conjugation as:

$h \in S_3$	s	s^2	t	ts	st
$h^{-1}\sigma_{\alpha,\beta}h$	$\sigma_{\beta-\alpha,-\alpha}$	$\sigma_{-\beta,\alpha-\beta}$	$\sigma_{-\alpha,\beta-\alpha}$	$\sigma_{\beta,\alpha}$	$\sigma_{\alpha-\beta,-\beta}$

Remark 2. An automorphism $\sigma : F_n \to F_n$ acts on functions $f \in k(F_n)$ by $\sigma(f) = f \circ \sigma^{-1}$. The group acts on the left on points, so $(\sigma_1\sigma_2)P = \sigma_1(\sigma_2 P)$, and the action on functions satisfies $(\sigma_1\sigma_2 f) = f \circ (\sigma_1\sigma_2)^{-1} = f \circ \sigma_2^{-1} \circ \sigma_1^{-1} = \sigma_1(\sigma_2 f)$.

In [11] and [18] the authors prove that F_n has genus $g = \frac{(n-1)(n-2)}{2}$, automorphism group $G = A \rtimes S_3$ and that the action of G on the function field $k(F_n)$, i.e., the field $k(x,y)$ subject to the equation $x^n + y^n + 1 = 0$, is given by

$g \in G$	$\sigma_{\alpha,\beta}$	s	s^2	t	ts	st
$g(x,y)$	$(\zeta_n^\alpha x, \zeta_n^\beta y)$	$\left(\frac{y}{x}, \frac{1}{x}\right)$	$\left(\frac{1}{y}, \frac{x}{y}\right)$	$\left(\frac{1}{x}, \frac{y}{x}\right)$	(y,x)	$\left(\frac{x}{y}, \frac{1}{y}\right)$

where ζ_n is a fixed primitive n-th root of unity. The above gives us the second required input item for Algorithm 2. Regarding the first, we use the character table of G that was computed in [1, Proposition 3]. Recall that S_3 has three irreducible representations: the trivial representation, the sign representation and the standard representation, denoted by $\rho_{\text{triv}}, \rho_{\text{sgn}}$ and ρ_{stan} respectively.

Proposition 1. *The irreducible representations of G are given in the table below,*

Rep.	***Degree***	***Character*** $\chi(\sigma_{\alpha,\beta}x)$, *where* $x \in S_3$
$\theta_{\frac{\nu n}{3},\frac{\nu n}{3},\rho}$	1	$\zeta^{\frac{\nu n}{3}(\alpha+\beta)}\chi_\rho(x)$
$\theta_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{stan}}}$	2	$\zeta^{\frac{\nu n}{3}(\alpha+\beta)}\chi_{\text{stan}}(x)$
$\theta_{\kappa,\kappa,\rho}$	3	$\begin{cases} \zeta^{\kappa(\alpha+\beta)} + \zeta^{\kappa(\alpha-2\beta)} + \zeta^{\kappa(\beta-2\alpha)} & , \text{ if } x = 1 \\ \zeta^{\kappa(\alpha+\beta)}\chi_\rho(x) & , \text{ if } x = ts \\ \zeta^{\kappa(\alpha-2\beta)}\chi_\rho(x) & , \text{ if } x = t \\ \zeta^{\kappa(\beta-2\alpha)}\chi_\rho(x) & , \text{ if } x = st \\ 0 & , \text{ if } x = s, s^2 \end{cases}$
$\theta_{\kappa,\lambda,\rho_{\text{triv}}}$	6	$\begin{cases} \begin{pmatrix} \zeta^{\kappa\alpha+\lambda\beta} + \zeta^{-(\kappa+\lambda)\alpha+\kappa\beta} + \zeta^{\lambda\alpha-(\kappa+\lambda)\beta} + \\ \zeta^{\lambda\alpha+\kappa\beta} + \zeta^{-(\kappa+\lambda)\alpha+\lambda\beta} + \zeta^{\kappa\alpha-(\kappa+\lambda)\beta} \end{pmatrix} & , \text{ if } x = 1 \\ 0 & , \text{ if } x \neq 1 \end{cases}$

where $\nu \in \{0,1,2\}$, $\rho \in \{\rho_{\text{triv}}, \rho_{\text{sgn}}\}$, $\kappa, \lambda \in \mathbb{Z}/n\mathbb{Z}$, $\kappa, \lambda \neq \frac{\nu n}{3}, \kappa \neq \lambda, \kappa \neq -2\lambda, \lambda \neq -2\kappa$ *and the representations corresponding to* $\kappa, \lambda \in \{\frac{n}{3}, \frac{2n}{3}\}$ *appear only when* $3 \mid n$.

In what follows, we fix all primitive roots of unity to be compatible with the chosen ζ_n, in the sense that if $n \mid i$, then ζ_i must satisfy $\zeta_i^{i/n} = \zeta_n$, whereas if $i \mid n$ then $\zeta_i = \zeta_n^{n/i}$.

Proposition 2. *The quotient F_n/G is isomorphic to $\mathbb{P}^1_k$, the branch locus of $F_n \to \mathbb{P}^1_k$ consists of three points P_∞, P_1, P_0. The points $Q_\infty = (\zeta_{2n}, 0), Q_1 =$*

$(1, \sqrt[n]{-2}), Q_0 = (\zeta_{6n}^4, \zeta_{6n}^2)$ of F_n lie above each of the three mentioned points, and their isotropy groups and monodromy generators are given in the following table.

point	*group*	*monodromy*
$Q_\infty = (\zeta_{2n}, 0)$	$\mathbb{Z}/2n\mathbb{Z}$	$\sigma_{1,1}t$
$Q_1 = (1, \sqrt[n]{-2})$	$\mathbb{Z}/2\mathbb{Z}$	t
$Q_0 = (\zeta_{6n}^4, \zeta_{6n}^2)$	$\mathbb{Z}/3\mathbb{Z}$	$\sigma_{-1,-1}s^2$

Proof. The proof can be found at the appendix.

The above implies that, in the notation of Theorem 3, we have $g_Y = 0$ and $|\mathscr{B}| = 3$. Thus the third and fourth term of $H_{S_X,V}$ simplify as follows

$$\frac{3T-1}{(1-T)^2}(g_Y - 1)\dim V + \frac{T}{(1-T)^2}\dim V|\mathscr{B}| = \frac{\dim V}{(1-T)^2}.$$

Theorem 4. *With the above notation, we have that*

$$H_{S_X,V}(T) = N_{V^*,1} + \delta_V T + \frac{\dim V}{(1-T)^2} - \frac{1}{1-T}\sum_Q \frac{f'_{Q,V}(1)}{e_Q} - \frac{T}{1-T}\sum_Q \frac{f_{Q,V}(T)}{1-T^{e_Q}},$$

where $\delta_V = 1$ for $V = V_{\mathrm{triv}}$ and 0 otherwise, the polynomials $f_{Q,V}(T)$ are given in the table below

$V \in \mathrm{Irr(G)}$	$f_{Q_\infty,V}(T)$	$f_{Q_0,V}(T)$	$f_{Q_1,V}(T)$
$\theta_{0,0,\rho_{\mathrm{triv}}}$	1	1	1
$\theta_{0,0,\rho_{\mathrm{sgn}}}$	T^n	1	T
$\theta_{\frac{n}{3},\frac{n}{3},\rho_{\mathrm{triv}}}$	$T^{\frac{4n}{3}}$	T	1
$\theta_{\frac{n}{3},\frac{n}{3},\rho_{\mathrm{sgn}}}$	$T^{\frac{n}{3}}$	T	T
$\theta_{\frac{2n}{3},\frac{2n}{3},\rho_{\mathrm{triv}}}$	$T^{\frac{2n}{3}}$	T^2	1
$\theta_{\frac{2\nu n}{3},\frac{2\nu n}{3},\rho_{\mathrm{sgn}}}$	$T^{\frac{5n}{3}}$	T^2	T
$\theta_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\mathrm{stan}}}$	$T^{\frac{\nu n}{3}} + T^{n+\frac{\nu n}{3}}$	$1+T+T^2-T^\nu$	$1+T$
$\theta_{\kappa,\kappa,\rho_{\mathrm{triv}}}$	$T^\kappa + T^{n+\kappa} + T^{[-2\kappa]_n}$	$1+T+T^2$	$2+T$
$\theta_{\kappa,\kappa,\rho_{\mathrm{sgn}}}$	$T^\kappa + T^{n+\kappa} + T^{[-2\kappa]_n+n}$	$1+T+T^2$	$1+2T$
$\theta_{\kappa,\lambda,\rho_{\mathrm{triv}}}$	$T^\kappa + T^\lambda + T^{n+\kappa} + T^{n+\lambda}$ $+T^{[-(\kappa+\lambda)]_n} + T^{[-(\kappa+\lambda)]_n+n}$	$2(1+T+T^2)$	$3+3T$

and $[x]_n$ denotes the smallest non-negative remainder of the division of x by n.

The proof of the above will be given separately for each of $f_{Q_\infty,V}$, $f_{Q_0,V}$, $f_{Q_1,V}$, by considering all irreducible representations of Proposition 1. To do so, one needs to calculate first the multiplicities $n_{d,Q,V}$, $Q \in \{Q_\infty, Q_0, Q_1\}$ as follows:

1. For each Q, write $G_Q = \{\sigma_Q^i : 0 \le i \le e_Q\}$ where σ_Q is the local monodromy and e_Q is the ramification index, both taken from Proposition 2.
2. For each $0 \le i \le e_Q$, find $\sigma_{\alpha_i,\beta_i} \in A$ and $x_i \in S_3$ such that $\sigma_Q^i = \sigma_{\alpha_i,\beta_i} x_i$.
3. Fix a primitive root of unity ζ_{e_Q} compatible with ζ_n as discussed above.
4. Compute $n_{d,Q,V} = \langle \mathrm{Res}^G_{G_Q}(\chi_V), \chi_Q^d\rangle = \sum_{i=0}^{e_Q-1} \chi_V\left(\sigma_P^i\right)\zeta_{e_Q}^{-id}$.

3.1 The Polynomials $f_{Q_\infty,V}(T)$

By Proposition 2, $Q_\infty = (\zeta_{2n}, 0)$ and G_{Q_∞} is generated by the monodromy element $\sigma_{1,1}t$. Since $(\sigma_{1,1}t)^2 = \sigma_{0,1}$, we have that

$$(\sigma_{1,1}t)^{2k} = \sigma_{0,k} \text{ and } (\sigma_{1,1}t)^{2k+1} = \sigma_{1,k+1}t, \qquad \text{for } 0 \le k \le n-1,$$

and thus, for $0 \le d \le 2n-1$, we have that

$$\begin{aligned} n_{d,Q_\infty,V} &= \langle \mathrm{Res}^G_{G_{Q_\infty}}(\chi_V), \chi^d_{Q_\infty}\rangle \\ &= \frac{1}{2n}\sum_{\ell=0}^{n-1} \zeta_{2n}^{2\ell(-d)}\chi_V(\sigma_{0,\ell}) + \frac{1}{2n}\sum_{\ell=0}^{n-1}\zeta_{2n}^{(2\ell+1)(-d)}\chi_V(\sigma_{1,\ell+1}t). \end{aligned}$$

- When $V = \theta_{\frac{\nu n}{3},\frac{\nu n}{3},\rho}$, $\rho \in \{\rho_{\mathrm{triv}}, \rho_{\mathrm{sgn}}\}$, Proposition 1 gives

$$\chi_V(\sigma_{0,\ell}) = \zeta_n^{\frac{\nu n \ell}{3}} = \zeta_{2n}^{\frac{2\nu n\ell}{3}} \text{ and } \chi_V(\sigma_{1,\ell+1}t) = \zeta_n^{\frac{\nu n(\ell+2)}{3}}\chi_\rho(t) = \zeta_{2n}^{\frac{2\nu n(\ell+2)}{3}}\chi_\rho(t).$$

Thus, we compute

$$\begin{aligned} n_{d,Q_\infty,V} &= \frac{1}{2n}\sum_{\ell=0}^{n-1}\zeta_{2n}^{2\ell(-d)}\zeta_{2n}^{\frac{2\nu n\ell}{3}} + \frac{1}{2n}\sum_{\ell=0}^{n-1}\zeta_{2n}^{(2\ell+1)(-d)}\zeta_{2n}^{\frac{2\nu n(\ell+2)}{3}}\chi_\rho(t) \\ &= \frac{1}{2n}\sum_{\ell=0}^{n-1}\zeta_{2n}^{2\ell(\frac{\nu n}{3}-d)} + \frac{1}{2n}\sum_{\ell=0}^{n-1}\zeta_{2n}^{2\ell(\frac{\nu n}{3}-d)+\left(\frac{4\nu n}{3}-d\right)}\chi_\rho(t) \\ &= \begin{cases} \frac{1}{2} + \frac{1}{2}\zeta_{2n}^{\frac{4\nu n}{3}-d}\chi_\rho(t) & \text{, if } n \mid \frac{\nu n}{3} - d \\ 0 & \text{, otherwise.} \end{cases} \end{aligned}$$

The only two values of d such that $0 \le d \le 2n-1$ and $n \mid \frac{\nu n}{3} - d$ are $d = \frac{\nu n}{3}$ and $d = n + \frac{\nu n}{3}$. So

$$f_{Q_\infty,V}(T) = \left(\frac{1}{2} + \frac{1}{2}\zeta_{2n}^{\nu n}\chi_\rho(t)\right)T^{\frac{\nu n}{3}} + \left(\frac{1}{2} + \frac{1}{2}\zeta_{2n}^{(\nu-1)n}\chi_\rho(t)\right)T^{n+\frac{\nu n}{3}}.$$

Notice that for $\nu \in \{0,1,2\}$ we always have that $\{\zeta_{2n}^{\nu n}, \zeta_{2n}^{(\nu-1)n}\} = \{-1, 1\}$.

- When $V = \theta_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\text{stan}}}$, Proposition 1 gives

$$\chi_V(\sigma_{0,\ell}) = 2\zeta_{2n}^{\frac{2\nu n\ell}{3}} \text{ and } \chi_V(\sigma_{1,\ell+1}t) = 0.$$

We compute as above

$$n_{d,Q_\infty,V} = \frac{1}{n}\sum_{\ell=0}^{n-1}\zeta_{2n}^{2\ell(\frac{\nu n}{3}-d)} + 0 = \begin{cases} 1 & , \text{if } n \mid \frac{\nu n}{3} - d \\ 0 & , \text{otherwise} \end{cases},$$

and thus

$$f_{Q_\infty,V}(T) = T^{\frac{\nu n}{3}} + T^{n+\frac{\nu n}{3}}.$$

- When $V = \theta_{\kappa,\kappa,\rho}$, $\rho \in \{\rho_{\text{triv}}, \rho_{\text{sgn}}\}$, Proposition 1 gives

$$\chi_V(\sigma_{0,\ell}) = 2\zeta_n^{\kappa\ell} + \zeta_n^{-2\kappa\ell} = 2\zeta_{2n}^{2\kappa\ell} + \zeta_{2n}^{-4\kappa\ell} \text{ and}$$
$$\chi_V(\sigma_{1,\ell+1}t) = \zeta_n^{-\kappa(2\ell+1)}\chi_\rho(t) = \zeta_{2n}^{-2\kappa(2\ell+1)}\chi_\rho(t).$$

We then have

$$\begin{aligned} n_{d,Q_\infty,V} &= \frac{1}{n}\sum_{\ell=0}^{n-1}\zeta_{2n}^{2\ell(\kappa-d)} + \frac{1}{2n}\sum_{\ell=0}^{n-1}\zeta_{2n}^{-2\ell(2\kappa+d)} + \frac{1}{2n}\sum_{\ell=0}^{n-1}\zeta_{2n}^{-2\ell(2\kappa+d)}\zeta_{2n}^{-(2\kappa+d)}\chi_\rho(t) \\ &= \begin{cases} 1 & , \text{if } n \mid \kappa - d \\ \frac{1}{2} + \frac{1}{2}\zeta_{2n}^{-(2\kappa+d)}\chi_\rho(t) & , \text{if } n \mid 2\kappa + d \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The first case gives $d = \kappa$ or $d = n + \kappa$, while the second gives $d = [-2k]_n$ or $d = n + [-2k]_n$ so

$$f_{Q_\infty,V}(T) = T^\kappa + T^{n+\kappa} + \left(\frac{1}{2} + \frac{1}{2}\chi_\rho(t)\right)T^{[-2\kappa]_n} + \left(\frac{1}{2} - \frac{1}{2}\chi_\rho(t)\right)T^{[-2\kappa]_n+n}.$$

- Finally, for $V = \theta_{\kappa,\lambda,\rho_{\text{triv}}}$ we have that

$$\chi_V(\sigma_{0,\ell}) = 2\zeta_{2n}^{2\lambda\ell} + 2\zeta_{2n}^{2\kappa\ell} + 2\zeta_{2n}^{-2(\kappa+\lambda)\ell} \text{ and } \chi_V(\sigma_{1,\ell+1}t) = 0.$$

Thus

$$\begin{aligned} n_{d,Q_\infty,V} &= \frac{1}{n}\sum_{\ell=0}^{n-1}\left(\zeta_{2n}^{2\ell(\lambda-d)} + \zeta_{2n}^{2\ell(\kappa-d)} + \zeta_{2n}^{-2\ell(\kappa+\lambda+d)}\right) \\ &= \begin{cases} 1 & , \text{if } n \mid \lambda - d \text{ or } n \mid \kappa - d \text{ or } n \mid \kappa + \lambda + d \\ 0 & , \text{otherwise} \end{cases} \end{aligned}$$

and so

$$f_{Q_\infty,V}(T) = T^\kappa + T^\lambda + T^{n+\kappa} + T^{n+\lambda} + T^{[-(\kappa+\lambda)]_n} + T^{[-(\kappa+\lambda)]_n+n}.$$

3.2 The Polynomials $f_{Q_0,V}(T)$

By Proposition 2, $Q_0 = (\zeta_{6n}^4, \zeta_{6n}^2)$ and G_{Q_0} is generated by the monodromy element $\sigma_{-1,-1}s^2$. For $d = 0, 1, 2$ we have

$$\begin{aligned} n_{d,Q_0,V} &= \langle \mathrm{Res}^G_{G_{Q_0}}(\chi_V), \chi^d_{Q_0} \rangle \\ &= \frac{1}{3}\left(\chi_V(1) + \zeta_3^{-d}\chi_V(\sigma_{-1,-1}s) + \zeta_3^{2(-d)}\chi_V\left((\sigma_{-1,-1}s^2)^2\right)\right). \end{aligned}$$

- For both $V = \theta_{\frac{\nu n}{3},\frac{\nu n}{3},\rho}$, $\rho \in \{\rho_{\mathrm{triv}}, \rho_{\mathrm{sgn}}\}$ we have

$$\begin{aligned} n_{d,Q_0,V} &= \frac{1}{3}\left(1 + \zeta_3^{-d}\zeta_n^{-\frac{2\nu n}{3}} + \zeta_3^{2(-d)}\zeta_n^{-\frac{4\nu n}{3}}\right) = \frac{1}{3}\left(1 + \zeta_3^{-d-2\nu} + \zeta_3^{-2d-4\nu}\right) \\ &= \begin{cases} 1 & \text{, if } d \equiv -2\nu \bmod 3 \equiv \nu \bmod 3 \\ 0 & \text{, otherwise.} \end{cases} \end{aligned}$$

and so $f_{Q_0,V}(T) = T^{[\nu]_3} = T^\nu$.

- When $V = \theta_{\frac{\nu n}{3},\frac{\nu n}{3},\rho_{\mathrm{stan}}}$ we have

$$\begin{aligned} n_{d,Q_0,V} &= \frac{1}{3}\left(2 - \zeta_3^{-d-2\nu} - \zeta_3^{-2d-4\nu}\right) = \frac{1}{3}\left(3 - 1 - \zeta_3^{-d-2\nu} - \zeta_3^{-2d-4\nu}\right) \\ &= \begin{cases} 1 & \text{, if } d \not\equiv -2\nu \bmod 3 \\ 0 & \text{, otherwise} \end{cases} \end{aligned}$$

and so $f_{Q_0,V}(T) = 1 + T + T^2 - T^{[-2\nu]_3} = 1 + T + T^2 - T^\nu$.

- When $V = \theta_{\kappa,\kappa,\rho}$, $\rho \in \{\rho_{\mathrm{triv}}, \rho_{\mathrm{sgn}}\}$, we have $n_{d,Q_0,V} = 1$ for $d \in \{0, 1, 2\}$ and so $f_{Q_0,V} = 1 + T + T^2$.
- When $V = \theta_{\kappa,\lambda,\rho_{\mathrm{triv}}}$, $n_{d,Q,V} = 2$ for $d \in \{0, 1, 2\}$ and so $f_{Q_0,V} = 2(1+T+T^2)$.

3.3 The Polynomials $f_{Q_1,V}(T)$

By Proposition 2, $Q_1 = (1, \sqrt[n]{-2})$ and G_{Q_1} is generated by the monodromy element t. For $d \in \{0, 1\}$ we have

$$\begin{aligned} n_{d,Q_1,V} &= \langle \mathrm{Res}^G_{G_{Q_1}}(\chi_V), \chi^d_{Q_1} \rangle \\ &= \frac{1}{2}\left(\chi_V(1) + (-1)^{-d}\chi_V(t)\right) = \frac{1}{2}\left(\dim V + (-1)^d\chi_V(t)\right) \end{aligned}$$

- When $V = \theta_{\frac{\nu n}{3},\frac{\nu n}{3},\rho}$, $\rho \in \{\rho_{\mathrm{triv}}, \rho_{\mathrm{sgn}}, \rho_{\mathrm{stan}}\}$, we get

$$f_{Q_1,V}(T) = \begin{cases} 1 & \text{, if } \rho = \rho_{\mathrm{triv}} \\ T & \text{, if } \rho = \rho_{\mathrm{sgn}} \\ 1 + T & \text{, if } \rho = \rho_{\mathrm{stan}}. \end{cases}$$

- When $V = \theta_{\kappa,\kappa,\rho}$, $\rho \in \{\rho_{\mathrm{triv}}, \rho_{\mathrm{sgn}}\}$, we get

$$f_{Q_1,V}(T) = \begin{cases} 2 + T & \text{, if } \rho = \rho_{\mathrm{triv}} \\ 1 + 2T & \text{, if } \rho = \rho_{\mathrm{sgn}}. \end{cases}$$

- Finally, when $V = \theta_{\kappa,\lambda,\rho_{\mathrm{triv}}}$, $f_{Q_1,V}(T) = 3 + 3T$.

4 Implementation and Examples

Let F_n be a Fermat curve over k, with the assumptions on n and k as in the previous section. By Petri's Theorem 1, there exists a short exact sequence

$$0 \to I_X := \ker \phi \hookrightarrow S := \operatorname{Sym}\left(H^0(X, \Omega_X)\right) \overset{\phi}{\twoheadrightarrow} S_X := \bigoplus_{m=0}^{\infty} H^0(X, \Omega_X^{\otimes m}) \to 0,$$

which is split over kG, since $\operatorname{char}(k) \nmid |G|$.

In this section, we present our Sage [16] program[3,4], which, as mentioned in the introduction, computes for each $V \in \operatorname{Irr}(G)$:

1. $H_{S_X,V}(T)$, by implementing the formulas of Theorem 4.
2. $H_{S,V}(T)$, by implementing Molien's formula (see Eq. 1).
3. $H_{I_X,V}(T) = H_{S,V}(T) - H_{S_X,V}(T)$, by subtracting the two above results.

The computation of $H_{S_X,V}(T)$ follows by Theorem 4. First we implement the difference $H_{S_X,V}(T) - N_{V^*,1} - \delta_V T$ for each $V \in \operatorname{Irr}(\mathrm{G})$. Then we read each N_{V^*1} from the implementation of $H_{S_X,V^*}(T) - N_{V,1} - \delta_{V^*} T$ and add T if V is trivial. We remark that implementing the algorithm for $n = 6$ we retrieve same results as in [1, Table 1, pg. 18], where we computed the kG-structure of $H^0(X, \Omega_X^{\otimes m})$ using an alternative approach. For example, the series for $V = \theta_{0,1,\mathrm{triv}}$ is

$$H_{S_X,V}(T) = \frac{T^3}{T^5 - 2\,T^4 + T^3 + T^2 - 2\,T + 1}.$$

To implement Molien's formula, it is required to input the character table of G and the representation $G \to \mathrm{GL}\left(H^0(X, \Omega_X)\right)$. The former is taken directly from Proposition 1, while the latter was implemented using the action of G on a basis $\{\omega_{i,j}\}$ of $H^0(X, \Omega_X)$ which we computed in [1, Prop. 6]:

$\sigma_{\alpha,\beta}(\omega_{i,j})$	$s(\omega_{i,j})$	$t(\omega_{i,j})$	$ts(\omega_{i,j})$	$st(\omega_{i,j})$	$s^2(\omega_{i,j})$
$\zeta^{\alpha(i+1)+\beta(j+1)}\omega_{i,j}$	$\omega_{n-3-(i+j),i}$	$-\omega_{n-3-(i+j),j}$	$-\omega_{j,i}$	$-\omega_{i,n-3-(i+j)}$	$\omega_{j,n-3-(i+j)}$

The output is much more complicated than $H_{S_X,V}(T)$: for instance when $n = 6$ and $V = \theta_{0,1,\mathrm{triv}}$ we obtain a rational function with numerator of degree 18 and denominator of degree 30.

The final step is to compute the equivariant Hilbert series of I_X using Petri's Theorem 1. For $n = 6$ and $V = \theta_{0,1,\mathrm{triv}}$, $H_{I_X,V}(T) = H_{S,V}(T) - H_{S_X,V}(T)$ has power series expansion

$$8\,T^3 + 20\,T^4 + 49\,T^5 + 130\,T^6 + 319\,T^7 + 667\,T^8 + 1363\,T^9 + 2557\,T^{10} + \text{higher order terms.}$$

The interpretation is that the representation $\theta_{0,1,\mathrm{triv}}$ appears, for example, 2557 times in the decomposition of the degree 10 graded piece of I_X into irreducible summands.

[3] http://users.uoa.gr/ kontogar/Code/EquivariantSage.ipynb.
[4] http://users.uoa.gr/ kontogar/Code/EquivariantSage.pdf.

5 Appendix - The Ramification Data of Fermat Curves

In this section we give the details for the proof of Proposition 2. We shall work over $k = \mathbb{C}$ for simplicity, even though the arguments are valid over any algebraically closed field of characteristic prime to the order of G. Recall that all roots of unity are fixed, as per the discussion preceding Proposition 2.

The Fermat curve can be seen as double Kummer cover of the projective line $\mathbb{P}^1$. We will work with Galois extensions of the corresponding function fields and in this way we have the Kummer extension of function fields $\mathbb{C}(F_n)/\mathbb{C}(x)$, where $\mathbb{C}(F_n)$ is the extension obtained by the rational function field $\mathbb{C}(x)$ by adjoining the quantity $y = (-1-x^n)^{\frac{1}{n}}$. Then we can consider the cyclic extension of function fields $\mathbb{C}(x)/\mathbb{C}(x^n)$. The ramification in such extensions is well known, see for example [9, 10], namely there is ramification in the cover $\mathbb{C}\left(F_n\right)/\mathbb{C}\left(F_n\right)^A$ over the points $x^n = -1, x^n = 0, x^n = \infty$, where $A = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

Since $G = \mathrm{Aut}(F_n) = A \rtimes S_3$, the Galois extension $\mathbb{C}(F_n)/\mathbb{C}(F_n)^G$ corresponding to the cover $F_n \to F_n/G$ has the intermediate subfield $\mathbb{C}(F_n)^A = \mathbb{C}(x^n)$, and $\mathbb{C}(F_n)^A/\mathbb{C}(F_n)^G$ is Galois with Galois group the symmetric group S_3. Moreover, the extension $\mathbb{C}(F_n)^A/\mathbb{C}(F_n)^G$ corresponds to a ramified cover $\mathbb{P}^1 \to \mathbb{P}^1$ ramified over three points. Such covers can be explained in terms of the j invariant, see [19]. Indeed, if we set $X = -x^n$ then the group S_3 can be realized by the six Möbius automorphisms:

$$X \mapsto \left\{X, \frac{1}{X}, 1-X, \frac{1}{1-X}, \frac{X}{1-X}, \frac{X-1}{X}\right\}.$$

The fixed points of these maps are given in the following table:

transform	order	equation	fixed points
$\frac{1}{X}$	2	$X^2-1=0$	$1, -1$
$1-X$	2	$2X-1$	$\frac{1}{2}, \infty$
$\frac{X}{X-1}$	2	$X^2-2X=0$	$0, 2$
$\frac{1}{1-X}, \frac{X-1}{X}$	3	$X^2-X+1=0$	$\zeta_6, \frac{1}{\zeta_6}$

and the function

$$j(X) = \frac{4}{27}\frac{(X^2-X+1)^3}{X^2(X-1)^2}$$

is a generator of the fixed field $\mathbb{C}(X)^{S_3} = \mathbb{C}(F_n)^G = \mathbb{C}(j)$. The fixed points of the S_3-cover $\mathbb{P}^1 \to \mathbb{P}^1$ are $P_{(j=0)}, P_{(j=1)}, P_{(j=\infty)}$. The map j maps the fixed points as follows:

X		$j(X)$
$0, 1, \infty$	$\longmapsto$	∞
$-1, 2, \frac{1}{2}$	$\longmapsto$	1
$\zeta_6, \frac{1}{\zeta_6}$	$\longmapsto$	0

In Fig. 1 we display the ramification diagram above the point $P_{(j=\infty)}$ and in Fig. 2 the respective diagram above the points $P_{(j=1)}$ and $P_{(j=0)}$. Note that in the first row we denote by $P_{i,i'}$ the i-th ramification point above $P_{(X=i')}$, for $i' \in \{0,1,\infty\}$, the labels in the vertical lines of the first column indicate the Galois groups, whereas in all other columns they indicate ramification indices.

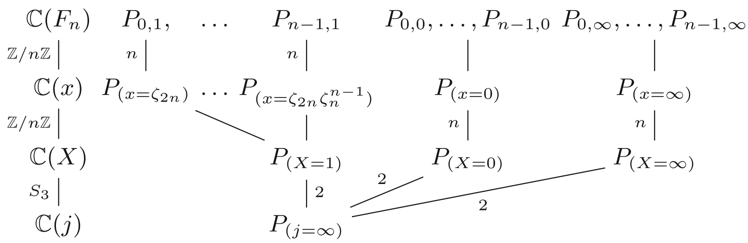

Fig. 1. Ramification diagram for $P_{(j=\infty)}$

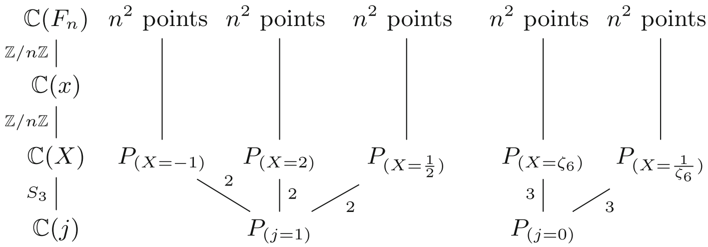

Fig. 2. Ramification diagram for $P_{(j=1)}$ and $P_{(j=\infty)}$

Each of the points $P_{(X=-1)}, P_{(X=2)}, P_{(X=\frac{1}{2})}, P_{(X=\zeta_6)}, P_{(X=\frac{1}{\zeta_6})}$ has n^2 points in the Fermat curve. For instance the point $X = -x^n = \zeta_6$ is lifted to the points (x,y) where $x = (-\zeta_6)^{1/n} = \zeta_n^{\ell}\zeta_{2n}\zeta_{6n} = \zeta_{6n}^{6\ell+4}$, for $0 \le \ell \le n-1$, and similarly, $y^n = -1 - x^n = -1 + \zeta_6 = \zeta_6^2$, since $\zeta_6^2 - \zeta_6 + 1 = 0$. Therefore, for $0 \le k \le n-1$, $y = \zeta_n^k \zeta_{6n}^2 = \zeta_{6n}^{6k+2}$. This means that the set of points $\{(\zeta_{6n}^{6\ell+4}, \zeta_{6n}^{6k+2}) : 0 \le k, \ell < n\}$ are the n^2 points above the point $P_{(X=\zeta_6)}$.

We will now select an arbitrary point above each $P_{(j=\infty)}, P_{(j=1)}, P_{(j=0)}$ and for each such point we will find the cyclic subgroup and the monodromy element. Recall that by Remark 2, automorphisms $\sigma \in G$ act on functions $f \in \mathbb{C}(F_n)$ by $\sigma(f) = f \circ \sigma^{-1}$.

• Consider the point $Q_\infty = (\zeta_{2n}, 0)$ above $P_{(j=\infty)}$. The isotropy subgroup is a cyclic group of order $2n$. For example we can verify that it is fixed by the element $\sigma_Q = \sigma_{1,1}t$. Further, since $(\sigma_{1,1}t)^2 = \sigma_{0,1}$, we have that

$$(\sigma_{1,1}t)^{2k} = \sigma_{0,k} \text{ and } (\sigma_{1,1}t)^{2k+1} = \sigma_{1,k+1}t, \qquad \text{for } 0 \le k \le n-1$$

A local uniformizer at Q_∞ is y, which is acted on by $(\sigma_{1,1}t)^2 = \sigma_{0,1}$ by $y \mapsto \zeta_n y$. Hence, the monodromy element at Q_∞ is $\sigma_{1,1}t$.

• Consider the point Q_1 above $P_{(j=1)}$ given by affine coordinates $(1, \sqrt[n]{-2})$, which is fixed by the automorphism t acting on functions as $t(x) = 1/x, t(y) = y/x$. Since the decomposition group at Q_1 is a cyclic group of order 2 the monodromy at Q_1 is the element t.

• A point $Q_0 = (x_0, y_0)$ above $P_{(j=0)}$ is given by $X = \zeta_6$, that is, $x_0^n = -\zeta_6$, therefore $x_0 = (-\zeta_6)^{1/n} = \zeta_n^\ell \zeta_{2n}\zeta_{6n} = \zeta_{6n}^{6\ell+4}$, for $0 \leq \ell \leq n-1$. Similarly $y_0^n = -1-x^n = -1+\zeta_6 = \zeta_6^2$, since $\zeta_6^2-\zeta_6+1 = 0$. Therefore, $y_0 = \zeta_n^k\zeta_{6n}^2 = \zeta_{6n}^{6k+2}$, for $0 \leq k \leq n-1$.

Let s be the automorphism acting on functions by $s(x) = y/x$, $s(y) = 1/x$, so that $s^2(x) = 1/y, s^2(y) = x/y$. Observe that the point with coordinates $(x_0, y_0) = \left(\zeta_{6n}^4, \zeta_{6n}^2\right)$ is sent by $\sigma_{1,1}s$ to (x_0, y_0). Indeed,

$$\left(\zeta_{6n}^4, \zeta_{6n}^2\right) \xmapsto{s} \left(\zeta_{6n}^{2-4}, \zeta_{6n}^{-4}\right) \xmapsto{\sigma_{1,1}} \left(\zeta_{6n}^4, \zeta_{6n}^2\right).$$

The function $x - x_0 = x - \zeta_{6n}^4$ is a local uniformizer at (x_0, y_0). By Remark 2, $\sigma_{1,1}s$ acts on functions as $\sigma_{-1,-1}s^2$ and thus

$$\begin{aligned}
\sigma_{-1,-1}s^2(x-\zeta_{6n}^4) &= \sigma_{-1,-1}\left(\frac{1}{y} - \zeta_{6n}^4\right) = \frac{\zeta_{6n}^6}{y} - \zeta_{6n}^4 = -\zeta_{6n}^4\frac{y-\zeta_{6n}^2}{y} \\
&= -\zeta_{6n}^4 \frac{y-\zeta_{6n}^2}{\zeta_{6n}^2 + (y-\zeta_{6n}^2)} \\
&= -\zeta_{6n}^2(y-\zeta_{6n}^2)\left(1 + \sum_{\nu=1}^{\infty}\frac{-1}{\zeta_{6n}^2}(y-\zeta_{6n}^2)^\nu\right).
\end{aligned} \tag{3}$$

On the other hand Taylor expansion of the Fermat equation at (x_0, y_0) gives

$$0 = x^n+y^n+1 = x_0^n+y_0^n+1+nx_0^{n-1}(x-x_0)+ny_0^{n-1}(y-y_0))+\text{higher order terms},$$

that is

$$\zeta_{6n}^{4(n-1)}(x-\zeta_{6n}^4) + \zeta_{6n}^{2(n-1)}(y-\zeta_{6n}^2) \mod \mathfrak{m}_{(x_0,y_0)}^2$$

and this combined with Eq. (3) gives

$$\sigma_{-1,-1}s^2(x-\zeta_{6n}^4) = \zeta_{6n}^2\zeta_{6n}^{2(n-1)}(x-\zeta_{6n}^4) = \zeta_3(x-\zeta_{6n}^4),$$

i.e., $\sigma_{-1,-1}s^2$ is indeed the monodromy at the point $Q_0 = (x_0, y_0)$.

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