Scalar-Gauss-Bonnet theories: evasion of no-hair theorems and novel black-hole solutions

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We consider a general Einstein–scalar–Gauss-Bonnet theory with a coupling function $f(\phi)$ between the scalar field and the quadratic gravitational Gauss-Bonnet term. We show that the existing no-hair theorems are easily evaded, and therefore black holes may emerge in the context of this theory. Indeed, we demonstrate that, under mild only assumptions for $f(\phi)$, asymptotic solutions describing either a regular black-hole horizon or an asymptotically-flat solution always emerge. We then show, through numerical integration, that the field equations allow for the smooth connection of these asymptotic solutions, and thus for the construction of a complete, regular black-hole solution with non-trivial scalar hair. We present and discuss the physical characteristics of a large number of such solutions for a plethora of coupling functions $f(\phi)$. Finally, we investigate whether pure scalar-Gauss-Bonnet black holes may arise in the context of our theory when the Ricci scalar may be altogether ignored.

Keywords: Generalised Gravitational Theories, Gauss-Bonnet term, no-hair theorems, black-hole solutions, scalar hair

1. Introduction

The General Theory of Relativity is a beautiful mathematical theory that predicts a variety of gravitational solutions, with the black holes being the most fascinating example. In the context of General Relativity, the black-hole solutions have been uniquely determined and classified according to their properties (mass, charge and angular-momentum). No-Hair theorems, that forbid the association of a black hole with any other "charge" or field, were formulated quite early on. The existence of black-hole solutions associated with a non-trivial scalar field in the region outside the black-hole horizon has also been intensively studied. The *old no-hair theorem*[1] was formulated in the seventies, and excluded static black holes with a scalar field. However, this was outdated by the discovery of black holes with Yang-Mills[2], Skyrme fields[3] or conformally-coupled scalar fields[4]. Twenty years later, the *novel no-hair theorem*[5] was formulated (for more recent analyses, see[6–8]) but this was also shown to be evaded in the context of the Einstein-Dilaton-Gauss-Bonnet theory [9] and in shift-symmetric Galileon theories [10,11].

In fact, the black-hole solutions [9–11] were derived in the context of the so-called generalised gravitational theories, where additional fields and higher gravitational terms may be present. These theories comprise a popular test-bed for the formulation of the ultimate theory of gravity beyond Einstein's General Theory of Relativity, and are under intense research activity. In this work, we will consider a wide class of gravitational theories where a scalar field ϕ has a general coupling function $f(\phi)$ to the quadratic gravitational Gauss-Bonnet (GB) term. Choosing the coupling function to be of an exponential or a linear form, one recovers the two novel black-hole solutions with non-trivial scalar hair [9,11], respectively. In [12] we demonstrated that, in fact, this class of theories with an arbitrary $f(\phi)$ always evades the existing no-hair theorems and allow for the emergence of novel black-hole solutions, with a regular horizon and an asymptotically-flat limit. Here, we review these results and discuss the characteristics of these solutions. We also investigate whether solutions arise in the context of the pure scalar-Gauss-Bonnet theory where the Ricci scalar may be ignored.

2. The Einstein-Scalar-Gauss-Bonnet theory

We will therefore consider the following generalised gravitational theory

$$S=\frac{1}{16\pi}\int d^4x\sqrt{-g}\left[R-\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi+f(\phi)R^2_{GB}\right], \tag{1}$$

where the GB term is defined as $R^2_{GB}=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2$. By varying the above action with respect to the metric tensor and scalar field, we obtain the following gravitational field equations and the equation for the scalar field:

$$G_{\mu\nu}=T_{\mu\nu}\,, \qquad \nabla^2\phi+\dot f(\phi)R^2_{GB}=0\,, \tag{2}$$

respectively, where a dot denotes the derivative with respect to the scalar field. The energy-momentum tensor has the form

$$T_{\mu\nu}=-\frac{1}{4}g_{\mu\nu}\partial_\rho\phi\partial^\rho\phi+\frac{1}{2}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}(g_{\rho\mu}g_{\lambda\nu}+g_{\lambda\mu}g_{\rho\nu})\eta^{\kappa\lambda\alpha\beta}\tilde R^{\rho\gamma}{}_{\alpha\beta}\nabla_\gamma\partial_\kappa f. \tag{3}$$

In the above, $\tilde R^{\rho\gamma}{}_{\alpha\beta}=\eta^{\rho\gamma\sigma\tau}R_{\sigma\tau\alpha\beta}=\epsilon^{\rho\gamma\sigma\tau}R_{\sigma\tau\alpha\beta}/\sqrt{-g}$. In the context of the above theory, we will look for regular, static, spherically-symmetric and asymptotically-flat black-hole solutions described by the line-element

$$ds^2=-e^{A(r)}dt^2+e^{B(r)}dr^2+r^2(d\theta^2+\sin^2\theta\,d\varphi^2)\,. \tag{4}$$

Using the above expression, the Einstein's equations take the following explicit form

$$4e^B(e^B+rB'-1)=\phi'^2\left[r^2e^B+16\ddot f(e^B-1)\right] -8\dot f\left[B'\phi'(e^B-3)-2\phi''(e^B-1)\right], \tag{5}$$

$$4e^B(e^B-rA'-1)=-\phi'^2r^2e^B+8\left(e^B-3\right)\dot fA'\phi', \tag{6}$$

$$e^B\left[rA'^2-2B'+A'(2-rB')+2rA''\right]=-\phi'^2re^B$$
$$+8\phi'^2\ddot{f}A'+4\dot{f}[\phi'(A'^2+2A'')+A'(2\phi''-3B'\phi')], \qquad (7)$$

while the scalar equation reads

$$2r\phi''+(4+rA'-rB')\,\phi'+\frac{4\dot{f}e^{-B}}{r}\left[(e^B-3)A'B'-(e^B-1)(2A''+A'^2)\right]=0. \qquad (8)$$

In the above, we have assumed that the scalar field depends only on the radial coordinate, and thus the prime denotes differentiation with respect to r.

The unknown quantities, that we seek to determine through the solution of the system of Eqs. (5)-(8), are the scalar field ϕ and the metric functions A and B. Of these, the metric function B may be easily determined in terms of (ϕ, A) through Eq. (6). Then, the remaining field equations lead to a system of two independent, ordinary differential equations of second order for the functions A and ϕ:

$$A''=\frac{P}{S}\,, \qquad \phi''=\frac{Q}{S}\,. \qquad (9)$$

The functions P, Q and S are rather complicated expressions of $(r,\phi',A',\dot{f},\ddot{f})$ and may be found in [12].

For a regular horizon to form, we demand that $e^A\to 0$ in Eq. (4), while ϕ, ϕ' and ϕ'' remain finite, as $r\to r_h$. Then, the 2nd of Eqs. (9) yields the constraint

$$\phi'_h=\frac{r_h}{4\dot{f}_h}\left(-1\pm\sqrt{1-\frac{96\dot{f}_h^2}{r_h^4}}\right). \qquad (10)$$

The quantity under the square-root should be positive which results in the additional bound $\dot{f}_h^2<r_h^4/96$. Using the above in the 1st of Eqs. (9), we may uniquely determine the form of A' near the horizon. Putting everything together, the near-horizon solution reads

$$e^A=a_1(r-r_h)+\dots, \qquad e^{-B}=b_1(r-r_h)+\dots,$$
$$\phi=\phi_h+\phi'_h(r-r_h)+\phi''_h(r-r_h)^2+\dots. \qquad (11)$$

On the other hand, at asymptotic infinity, we assume power-law expressions for the metric functions and scalar field as customary. Substituting these expressions into the field equations, we obtain

$$e^A=\;1-\frac{2M}{r}+\frac{MD^2}{12r^3}+\dots, \quad e^B=\;1+\frac{2M}{r}+\frac{16M^2-D^2}{4r^2}+\dots,$$
$$\phi=\;\phi_\infty+\frac{D}{r}+\frac{MD}{r^2}+\frac{32M^2D-D^3}{24r^3}+\dots. \qquad (12)$$

The above asymptotic behaviour is characterised by the ADM mass M and scalar charge D of the black hole. We may therefore conclude that the scalar-tensor theory

(1) with a general coupling function $f(\phi)$ is always compatible with either a regular horizon or an asymptotically-flat limit.

However, no complete black-hole solution may be constructed unless the aforementioned asymptotic solutions are smoothly matched. To investigate whether this is in principle possible, we turn to the novel no-hair theorem [5] and examine its requirements under which it may forbid the existence of such a solution. This theorem assumes first that, at asymptotic infinity, the $T^r{}_r$ component of the energy-momentum tensor is positive and decreasing. Indeed, we find that this has the form

$$T^r{}_r = \frac{e^{-B}\phi'}{4}\Big[\phi' - \frac{8e^{-B}\left(e^B-3\right)\dot{f}A'}{r^2}\Big] \simeq \frac{\phi'^2}{4} \sim \mathcal{O}(\frac{1}{r^4}). \tag{13}$$

In the near-horizon regime, $T^r{}_r$ should be negative and increasing according to [5]; however, employing the asymptotic solution (11), we find that in our case

$$T^r{}_r = -\frac{2e^{-B}}{r^2}A'\phi'\dot{f} + \mathcal{O}(r-r_h)\,. \tag{14}$$

This expression is always positive-definite since, close to the horizon, $A' > 0$, and $\dot{f}\,\phi' < 0$ according to Eq. (10) for a regular horizon. Also, we find that $T^r{}_r$ is always decreasing close to r_h and as a result, the novel no-hair theorem is non-applicable in our theory.

The above result opens the way for the construction of novel black-hole solutions in the context of the general theory (1). We have therefore numerically solved the system of equations (9), and determined a large number of black-hole solutions with scalar hair for a variety of forms of the coupling function $f(\phi)$: exponential, odd and even power-law, odd and even inverse-power-law. Once the form of $f(\phi)$ was chosen, the input values (ϕ_h, ϕ'_h), with ϕ'_h being given by Eq. (10), always led to a regular black-hole solution with scalar hair. The scalar field and profile of $T^r{}_r$ for those solutions are depicted in Figs. 1(a,b).

Some of the characteristics of the black-hole solutions we found [12] are represented in Figs. 2(a,b), where we depict the indicative case of $f(\phi) = \alpha/\phi$. The scalar charge D is a function of the black-hole mass and thus a dependent quantity; this renders

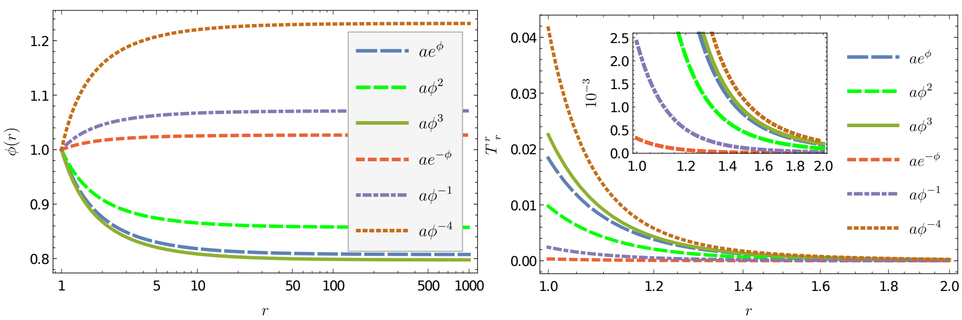

Fig. 1. The scalar field ϕ (left plot) and the $T^r{}_r$ component (right plot) for different coupling functions $f(\phi)$, for $a = 0.01$ and $\phi_h = 1$.

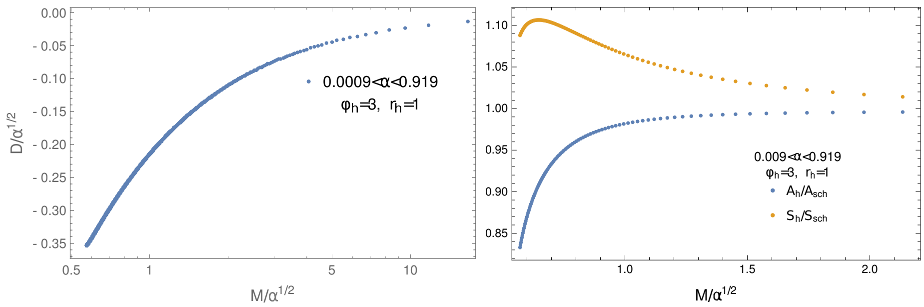

Fig. 2. The scalar charge D (left plot), and the ratios A_h/A_{Sch} and S_h/S_{Sch} (right plot, lower and upper curve respectively) in terms of the mass M, for $f(\phi) = \alpha/\phi$.

the scalar hair secondary. For a large mass, the scalar charge vanishes and our black-hole solutions match the Schwarzschild solution. The horizon area is always smaller than the one of the Schwarzschild solution exhibiting also a lower value beyond which the black hole ceases to exist — the latter feature is due to the additional bound emerging from the positivity of the quantity under the square-root in Eq. (10). Its entropy is larger than that of the Schwarzschild case and thus thermodynamically more stable.

3. The pure scalar-Gauss-Bonnet theory

We will now investigate whether a regular black-hole solution can arise in the context of a pure scalar-GB theory, i.e. in the absence of the linear Ricci term. By ignoring all terms in the field equations related to the Ricci term, these are simplified — but can we construct again a regular horizon? If we assume as before that, as $r \to r_h$, ϕ' remains finite while A' diverges, Eq. (6) now yields: $e^B \simeq 3 + \mathcal{O}(1/A')$; but this does not describe a black hole. We may alternatively demand that $e^B \to \infty$ instead, as $r \to r_h$; then, Eq. (6) gives: $A' \simeq r^2\phi'/8\dot{f} + \mathcal{O}(e^{-B})$. In this case, $A(r)$ is the dependent quantity, and Eqs. (5) and (7) form a system of two differential equations for B and ϕ. In the limit $r \to r_h$, we find the results[12]

$$B' = -\frac{2}{r}\, e^B + \mathcal{O}\left(e^{-B}\right), \qquad \phi'' = -\frac{e^B}{r}\, \phi' + \mathcal{O}\left(e^{-B}\right). \tag{15}$$

Upon integration, the first equation leads to the solution $e^{-B} = 2\ln\left(r/r_h\right)$, which does resemble a horizon, but the second one reveals that this horizon is not regular unless $\phi'(r_h) = 0$, an assumption that trivialises the contribution of the GB term. Alternative ansatzes for the form of the spacetime around the sought-for black hole have also failed to lead to a regular horizon in the absence of the Ricci scalar[15].

4. Conclusions

In the context of a general Einstein-scalar-GB theory, we have demonstrated that the emergence of regular black-hole solutions is a generic feature. For an arbitrary coupling function $f(\phi)$, we were always able to construct a regular black-hole horizon as well as an asymptotically-flat solution at infinity, and to explicitly show that the novel no-hair theorem is then easily evaded. Our numerical analysis has subsequently led to a large number of regular black-hole solutions for different choices of $f(\phi)$, all characterised by a non-trivial scalar hair (for similar black-hole solutions, see also[13,14]). The study of the pure scalar-GB theory, and the failure to obtain a regular horizon, clearly demonstrates that the presence of the GB term in the theory is a necessary condition for the emergence of novel black holes but not a sufficient one as it must be supplemented by the presence of the linear Ricci term.

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