A sustainable inventory policy for two substitutable products

Iris-Pandora Krommyda
Department of Mathematics, University of Ioannina 45110 Ioannina - Greece

Vasileios A. Tatsis
Department of Computer Science Engineering, University of Ioannina 45110 Ioannina - Greece

Konstantina Skouri
Department of Mathematics, University of Ioannina 45110 Ioannina - Greece

Abstract

We study an inventory problem considering two products with a fixed shelf life. The products are ordered simultaneously from the same supplier. We assume that their demand is freshness dependent, so in order to increase sales the retailer offers a single markdown near their expiration date. We also assume that the products are substitutable, so in case of a stock-out for one of the products, a known fraction of its demand can be satisfied by using the stock of the other product. In this context, our model can be applied to the management of foodstuffs. Hence, in accordance with EU guidelines for food waste reduction, we assume that unsold items at the end of the replenishment cycle can be donated to non-profit organizations or be sold at a salvage price to a secondary market. Due to space limitations, only numerical examples under different parametric scenarios are presented in order to illustrate the optimal policy that maximizes the retailer's profit.

KEYWORDS
Inventory management; Substitution; Perishability; Food waste;

1. INTRODUCTION

Products with a fixed shelf life constitute a large part of retail inventories. Previous studies usually assume that the demand for such products is dependent on the time remaining to their expiration date, i.e. freshness level ([1], [2]), including sometimes also price dependency ([3],[4],[5]) and/or stock dependency ([6],[7]).In the model, presented in this study, freshness dependent demand is also assumed. However, in order to be in accordance with the common practice of many retail stores, we assume that a single price markdown is offered when a product is close to its expiration date, in order to boost sales. This is also noted by [8], in their study of the effect of expiration dates on the purchasing behavior for grocery store perishables, that an effective mean of selling aging inventory is to offer a price discount near the expiration date of the product. Relevant papers include [9], [10] and [11].

In addition, we assume that the products are ordered simultaneously from the same supplier. This implies that the products are similar and therefore substitutable, such as different types of yogurt, milk, etc. So in case of a stock-out for one of the products, a known fraction of its demand can be satisfied by using the stock of the other product (see [12], [13]).

Furthermore, in accordance with EU guidelines for food waste reduction, we assume that unsold items at the end of the replenishment cycle can be donated to non-profit organizations or be sold at a salvage price to a secondary market. The only work we found regarding food donations in an EOQ context is the one by [14], who study a supply chain composed by a retailer and potential recipients of food recovery. They assume that the retailer deals with k products and maximize the joint profit of the supply chain by assuming that the demand of each product is linearly dependent on the time remaining to its expiration date and tax deductions are granted to the retailer due to the donation of surplus food.

The model presented in this work incorporates the above mentioned assumptions describing a situation confronted by many retailers. Our goal is to derive the optimal solution that maximizes the retailer's profit and at the same time reduces food waste. The rest of the paper is organized as follows. First, the notations and basic assumptions of the model are provided. Then the model formulation, for the different cases that arise, is presented. Due to space restrictions, the optimization procedure is omitted. However, numerical examples under different parametric scenarios are conducted, in order to illustrate the optimal solution to the problem.

2. NOTATIONS AND ASSUMPTION

To develop the mathematical model, the following notations and assumptions are used:

Notations (i=1,2):

D_i	demand for product *i*, per time unit	l_i	lost sales cost for product *i*, per unit
h_i	inventory holding cost for product *i*, per unit, per time unit, including capital cost	δ_{ij}	the percentage of demand that can be satisfied by product *j* in case of a stock out of product *i*
c_i	purchase cost for product *i*, per unit	W	total storage capacity for both products
p_i	selling price for product *i*, per unit	e_i	fixed shelf life of product *i*
β	discount percentage on selling price ($0 \le \beta \le 1$)	T	length of the replenishment cycle-reorder interval (decision variable)
p_i'	discounted selling price for product *i*, per unit $p_i' = (1-\beta)p_i$	t_i	the time at which a price markdown is offered for product *i* (decision variable)
A	joint ordering cost, per order	q_i	the inventory level of product *i* at time T (decision variable)
γ_1	opportunity gain due to the donation of food to non-profit organizations, per unit donated	Q_i	order quantity for product *i*
γ_2	selling price per unit of expired product to livestock market		

Assumptions:

1. The retailer deals with two products that have a fixed Shelf Life (SL), i.e. the products remain safe and suitable for human consumption until the reaching of the SL.
2. The planning horizon and replenishment rate are infinite.
3. The products are ordered simultaneously from the same supplier with a joint ordering cost.
4. The retailer's total storage capacity for both products is constant and equal to W. Obviously the order quantity cannot exceed capacity, i.e. $Q_1 + Q_2 \le W$.
5. Initially, the demand of each product is constant and equal to D_i. At time t_i the retailer offers a price markdown $p_i' = (1-\beta)p_i$, because product *i* is close to its expiration date. This increases the demand to $\alpha D_i, \alpha > 1$. However, at the same time, the customer becomes aware of the expiration date of the product and the demand becomes a decreasing function with respect to the time remaining before the expiration date. Hence, the demand of each product is defined as:
$$D_i(t) = \begin{cases} D_i, & 0 \le t < t_i \\ \alpha\left(\frac{e_i - t}{e_i}\right) D_i, & t_i \le t \le e_i \end{cases}.$$
6. At time T, the retailer withdraws the remaining products from the shelf and a new order arrives. We distinguish the following cases:
   - When $T < e_i$, the withdrawn items for product *i* are donated to a non-profit organization, this works as advertisement and creates a gain of goodwill towards the retailer from the customers which is quantified as a profit per unit of product donated (γ_1).
   - When $T = e_i$, the withdrawn items for product *i* are sold to a secondary market at a salvage price (γ_2).
   - When $T > e_i$, then during time period $e_i - T$ shortages occur for product *i*.
7. The products are substitutable, so the demand of product *i* during stock out can be satisfied by the other product at a known percentage $0 < \delta_{ij} \le 1$. The unsatisfied demand is completely lost.
8. It is realistic to assume that $p_i > p_i' > c_i$ and $\gamma_2 < c_i$.

3. MODEL FORMULATION

Without loss of generality, $e_1 < e_2$ is assumed. At time 0 a new order of $Q_1 + Q_2$ units arrives. During time period $[0, T]$ the inventory level of each product depletes due to demand $D_i(t)$, as defined in the assumptions. At time *T*, the retailer withdraws the remaining products from the shelf and a new order arrives. In order to formulate the mathematical model we distinguish the following cases depending on the order of the variables t_1, t_2, T and the parameters e_1, e_2:
$0 \le t_i \le T \le e_1 < e_2, i = 1,2$ (Case A),
$0 \le t_i \le e_1 \le T \le e_2$ $i = 1,2$ (Case B),
$0 \le t_1 \le e_1 \le t_2 \le T \le e_2$ (Case C).

Case A

The differential equations that describe the depletion of the inventory level for both products (*i*=1,2), for Case A, are as follows:

$$\frac{dI_i(t)}{dt} = -D_i,\ 0 \le t < t_i,$$
$$\frac{dI_i(t)}{dt} = -\alpha D_i\left(\frac{e_i - t}{e_i}\right),\ t_i \le t \le T,$$
$$I_i[T] = q_i,\ I_i[t_i^+] = I_i[t_i^-]$$

The retailer's profit per time unit, for this case, can generally be expressed as:

$$\Pi_A(T,t_1,t_2,q_1,q_2)=\begin{cases}\frac{1}{T}\{\overline{\pi}_A(T,t_1,t_2,q_1,q_2)+\gamma_1 q_1+\gamma_1 q_2\},\ 0\le t_i\le T<e_1<e_2,\ i=1,2\\ \frac{1}{T}\{\overline{\pi}_A(T,t_1,t_2,q_1,q_2)+\gamma_2 q_1+\gamma_1 q_2\},\ T=e_1,\ 0\le t_i\le e_1<e_2,\ i=1,2\end{cases},$$

where

$$\overline{\pi}_A(T,t_1,\ t_2,\ q_1,\ q_2) = p_1 D_1 t_1 + p_1'\int_{t_1}^{T}\alpha D_1\left(\frac{e_1-t}{e_1}\right)dt - c_1 Q_1 - h_1\int_0^{t_1} I_1(t)dt - h_1\int_{t_1}^{T} I_1(t)dt$$
$$+p_2 D_2 t_2 + p_2'\int_{t_2}^{T}\alpha D_2\left(\frac{e_2-t}{e_2}\right)dt - c_2 Q_2 - h_2\int_0^{t_2} I_2(t)dt - h_2\int_{t_2}^{T} I_2(t)dt - A.$$

The terms appearing in $\overline{\pi}_A(T,t_1,t_2,q_1,q_2)$ correspond to the sales revenue during the time each product has its original price, the sales revenue after the discount is offered, the purchasing cost and the holding cost for both products, as well as the joint ordering cost. The term $\gamma_1 q_i$ corresponds to the profit generated by the donation of the surplus quantity of product *i*, while the term $\gamma_2 q_i$ by selling the leftover products of product *i* to the secondary market.

The problem to be solved is:

$$\max \Pi_A(T,t_1,t_2,q_1,q_2)$$
$$\text{s.t. } 0 \le t_i \le T \le e_1 < e_2$$
$$q_i \ge 0,\ i = 1,2$$
$$Q_1 + Q_2 \le W$$

Note: Since the two products are independent, the order of t_1, t_2 will be derived by the optimal solution.

Case B

The differential equations that describe the depletion of the inventory level for both products (*i*=1,2), for Case B, are as follows:

$$\frac{dI_1(t)}{dt} = -D_1,\ 0 \le t < t_1,$$
$$\frac{dI_1(t)}{dt} = -\alpha D_1\left(\frac{e_1 - t}{e_1}\right),\ t_1 \le t \le e_1,$$
$$\frac{dI_1(t)}{dt} = -(1-\delta_{12})D_1,\ e_1 \le t < T,$$
$$I_1[e_1] = 0,\ I_1[t_1^+] = I_1[t_1^-],\ I_1[e_1^+] = I_1[e_1^-]$$

$$\frac{dI_2(t)}{dt} = -D_2,\ 0 \le t < t_2,$$
$$\frac{dI_2(t)}{dt} = -\alpha D_2\left(\frac{e_2 - t}{e_2}\right),\ t_2 \le t \le e_1,$$
$$\frac{dI_2(t)}{dt} = -\alpha D_2\left(\frac{e_2 - t}{e_2}\right) - \delta_{12}D_1,\ e_1 \le t \le T,$$
$$I_2[T] = q_2,\ I_2[t_2^+] = I_2[t_2^-],\ I_2[e_1^+] = I_2[e_1^-]$$

Note: Since we have assumed that $\gamma_2 < c_1$, there is obviously no point in having leftover quantity at the time of the expiration date of product 1 (the holding cost of keeping the extra inventory is greater than the gain of selling it to the livestock market). Hence, we set $I_1[e_1] = 0$.

The retailer's profit per time unit, for Case B, can generally be expressed as:

$$\Pi_B(T,t_1,t_2,q_2) = \begin{cases}\frac{1}{T}\{\overline{\pi}_B(T,t_1,t_2,q_2) - l_1(1-\delta_{12})D_1(T-e_1) + \gamma_1 q_2\},0 \le t_i \le e_1 \le T < e_2,\ i=1,2\\ \frac{1}{T}\{\overline{\pi}_B(T,t_1,t_2,q_2) - l_1(1-\delta_{12})D_1(T-e_1) + \gamma_2 q_2\},T = e_2,\ 0 \le t_i \le e_1,\ i=1,2\end{cases},$$

where

$$\overline{\pi}_B(T,t_1,t_2,q_2) = p_1 D_1 t_1 + p_1'\int_{t_1}^{e_1}\alpha D_1\left(\frac{e_1-t}{e_1}\right)dt - c_1 Q_1 - h_1\int_0^{t_1} I_1(t)dt - h_1\int_{t_1}^{e_1} I_1(t)dt + p_2 D_2 t_2$$
$$+p_2'\int_{t_2}^{T}\alpha D_2\left(\frac{e_2-t}{e_2}\right)dt + p_2'\delta_{12}D_1(T-e_1) - c_2 Q_2 - h_2\int_0^{t_2} I_2(t)dt - h_2\int_{t_2}^{e_1} I_2(t)dt - h_2\int_{e_1}^{T} I_2(t)dt - A.$$

Obviously, the term $l_1(1-\delta_{12})D_1(T-e_1)$ corresponds to the lost sales cost due to unsatisfied demand for product 1 during time period $T - e_1$.

The problem to be solved is:

$$\begin{gathered} max\, \Pi_B(T, t_1, t_2, q_2) \\ s.t.\, 0 \leq t_i \leq e_1 \leq T \leq e_2 \\ q_2 \geq 0,\ Q_1 + Q_2 \leq W \end{gathered}$$

Case C

The differential equations that describe the depletion of the inventory level, for Case C, are as follows:
For product 1 see case B.
For product 2:

$$\begin{gathered} \frac{dI_2(t)}{dt} = -D_2,\ 0 \leq t < e_1, \\ \frac{dI_2(t)}{dt} = -D_2 - \delta_{12} D_1,\ e_1 \leq t < t_2, \\ \frac{dI_2(t)}{dt} = -\alpha D_2 \left(\frac{e_2 - t}{e_2}\right) - \delta_{12}\, D_1,\ t_2 \leq t \leq T, \\ I_2[T] = q_2,\ I_2[t_2^+] = I_2[t_2^-],\ I_2[e_1^+] = I_2[e_1^-] \end{gathered}$$

The retailer's profit per time unit, for Case C, can generally be expressed as:

$$\Pi_C\big(T, t_1, t_2, q_2\big) = \begin{cases} \frac{1}{T}\{\pi_C\big(T, t_1, t_2, q_2\big) - l_1(1-\delta_{12})D_1(T-e_1) + \gamma_1 q_2\}, & 0 \leq t_1 \leq e_1 \leq t_2 \leq T < e_2 \\ \frac{1}{T}\{\pi_C\big(T, t_1, t_2, q_2\big) - l_1(1-\delta_{12})D_1(T-e_1) + \gamma_2 q_2\}, & T = e_2, 0 \leq t_1 \leq e_1 \leq t_2 \leq e_2 \end{cases},$$

where

$$\Pi_C\big(T,\ t_1,\ t_2,\ q_2\big) =$$

$$p_1 D_1 t_1 + p_1' \int_{t_1}^{e_1} \alpha\ D_1 \left(\frac{e_1 - t}{e_1}\right) dt - c_1 Q_1 - h_1 \int_0^{t_1} I_1(t) dt - h_1 \int_{t_1}^{e_1} I_1(t) dt + p_2 \delta_{12} D_1 (t_2 - e_1) + p_2 D_2 t_2$$
$$+ p_2' \delta_{12} D_1 (T - t_2) + p_2' \int_{t_2}^{T} \alpha D_2 \left(\frac{e_2 - t}{e_2}\right) dt - c_2 Q_2 - h_2 \int_0^{e_1} I_2(t) dt - h_2 \int_{e_1}^{t_2} I_2(t) dt - h_2 \int_{t_2}^{T} I_2(t) dt - A.$$

The problem to be solved is:

$$\begin{gathered} max\, \Pi_C(T, t_1, t_2, q_2) \\ s.t.\, 0 \leq\ t_1\ \leq\ e_1\ \leq\ t_2\ \leq\ T\ \leq\ e_2 \\ q_2 \geq\ 0,\ Q_1 + Q_2\ \leq\ W \end{gathered}$$

4. NUMERICAL EXAMPLES

In *order* to illustrate the optimal solution to the problem, numerical examples are presented under four different scenarios. The following parametric values are used: D_1=60, D_2=40, h_1=0.8, h_2=1.2, p_1=4, p_2=6, β=0.3, c_1=2, c_2=3, A=250, γ_1=2.5, γ_2=1, e_1=4, e_2=6, α=1.5, W=500, l_1=0.5, δ_{12}=0.3.

Scenario 1: Using the above default values of the parameters, the optimal solution to the problem is obtained by the first branch of profit function Π_A, i.e. t_1=t_2=T=2.28, q_1=q_2=0, Q_1=136.93, Q_2=91.29, with corresponding optimal profit Π_A=20.9. This solution indicates that for these specific parametric values it is not optimal to markdown the products during the replenishment cycle, or to keep leftover quantity at the end. The optimal cycle length is derived by the EOQ formula.

Scenario 2: We now lower the shelf life of product 1, to e_1=2. In this case the optimal solution is t_1=t_2=T=2, q_1=q_2=0, Q_1=120, Q_2=80, Π_A=19. This solution is obtained by the second branch of profit function Π_A, which indicates that the replenishment cycle of the retailer should coincide with the shelf life of product 1, no markdown should be offered and no leftover quantity should remain at the end of the cycle.

Scenario 3: As a third scenario, we assume a very high ordering cost, i.e. A=100 and a very low holding cost, as well as purchasing cost for the second product, i.e. h_2=0.1, c_2=1. The optimal solution, derived by the second branch of profit function Π_C, is t_1=3.17, t_2=T=6, q_2=0, Q_1=197.95, Q_2=276, Π_C=78.68. In this case, it is optimal for the retailer to coincide his replenishment cycle with the shelf life of the second product and to allow shortages for product 1. Also, a markdown should be offered for product 1 near its expiration date. Keeping leftover quantity of product 2 at the end of the cycle is not in the best interest of the retailer.

Scenario 4: Finally, in the fourth scenario, we assume a smaller ordering cost, i.e. A=100. In this case, the optimal solution derived is t_1=t_2=T=0.1, q_1=490, q_2=0, Q_1=496, Q_2=4, with corresponding optimal profit Π_A=1293.2. For this case, we observe that it is optimal to make the cycle length as small as possible (in order to be realistic, we assume that there exists a minimum reorder interval T_{min}=0.1) and to donate as much quantity as possible of product 1.The storage capacity is therefore fully utilized.