Reflecting on a series of studies on conceptual and procedural knowledge of fractions: Theoretical, methodological and educational considerations

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We present an overview of four studies investigating secondary students' conceptual and procedural knowledge of fractions qualitatively as well as quantitatively. We draw on these studies and their results to discuss the problem of measuring conceptual and procedural knowledge; the issue of individual differences in conceptual and procedural fraction knowledge; and related educational implications.

Keywords: Fractions, conceptual/procedural knowledge, individual differences, instruction.

Conceptual and procedural knowledge in mathematics learning

The distinction between procedural and conceptual knowledge has elicited considerable research and discussion among researchers in the fields of cognitive-developmental psychology and mathematics education. Procedural knowledge is typically defined as the ability to execute action sequences to solve problems and is usually tied to specific problem types, whereas conceptual knowledge is defined as knowledge of concepts pertaining to a domain and related principles (Rittle-Johnson & Schneider, 2015). This definition has been contested on the grounds that it depicts a very narrow picture for procedural knowledge as something that one either has, or does not have, ignoring different qualities that such knowledge may have (Star, 2005). Still, this definition captures the distinction between carrying out a procedure that pertains to a domain, and understanding of the entities (e.g., mathematical objects and mathematical relations) that are involved in the procedure. This distinction is valuable, particularly from the point of view of mathematics education that has been tantalized by the phenomenon of procedural skill without understanding before acknowledging that conceptual and procedural knowledge are equally important for students' mathematical development (e.g., De Corte, 2004).

The relation between the two types of knowledge has been difficult to establish empirically, in particular with respect to the order of acquisition (i.e., which type of knowledge develops first).The current prevailing model is the *iterative model* that assumes that there is a bi-directional relation between conceptual and procedural knowledge and accounts for many empirical findings, notably the well-documented finding that the two types of knowledge are typically highly correlated (Rittle-Johnson & Schneider, 2015). However, such correlations found at group level do not accurately reflect what happens at the individual level. Indeed, research shows that there are individual differences in the way students combine the two kinds of knowledge putting a challenge to the iterative model (e.g., Canobi, 2004; Hallett, Nunes, & Bryant, 2010; Hallett, Nunes, Bryant, & Thrope, 2012). In particular, Hallett and colleagues (2010, 2012) were the first to systematically study such individual differences in the area of fraction learning. They assessed conceptual and procedural knowledge of students at Grade 4 and 5 (2010) as well as at Grade 6 and 8 (2012) and they identified groups of students who were either strong or weak in both types of knowledge. However, they also consistently traced two substantial groups of students for whom there was discrepancy between the two types of knowledge (i.e., students in one group exhibited stronger procedural fraction knowledge, compared to their conceptual knowledge, and vice versa for students in the other group). Hallett and colleagues found that such discrepancies became less salient with age.

Before more progress can be made in understanding the relations between conceptual and procedural knowledge, more attention should be paid to the validity of measures of conceptual and procedural knowledge (Rittle-Johnson & Schneider, 2015). This is a challenging endeavor, particularly with respect to conceptual knowledge, which is considered a multi-dimensional construct; and it becomes even more challenging when it comes to the concept of fraction, itself a multifaceted construct pertaining to a vast variety of situations (Moss, 2005).

In this paper we present an overview of four studies investigating secondary students' conceptual and procedural knowledge of fractions qualitatively as well as quantitatively. We reflect on the challenges we met and we draw on the findings to discuss the problem of measuring conceptual and procedural knowledge; the issue of individual differences in conceptual and procedural fraction knowledge; and related educational implications.

Conceptual and procedural fraction knowledge of Greek secondary students

Study 1: General trends

In this study (Bempeni & Vamvakoussi, 2014) we looked at the development of conceptual knowledge of fractions in the first three years of secondary level (7^{th} to 9^{th} grade in Greece). We administered an open-ended questionnaire to 80 seventh and ninth graders. The questionnaire consisted of 24 items targeting many aspects of fraction knowledge. The tasks included interpreting and constructing fraction representations using the area model and the number line; fraction magnitude estimation and estimation of outcomes of arithmetical operations; fraction comparison tasks that could be tackled with conceptual strategies (e.g., comparing 3/2 and 13/27, for which 1 can be used as benchmark); tasks targeting the inappropriate transfer of natural number knowledge to fractions (e.g., "how many numbers are there between 2/5 and 3/5?"); and contextualized problems where physical quantities were involved, requiring understanding of the unit of reference(see Task 1 in Figure 1 for an example of the latter, albeit in multiple-choice format). The difficulty of items varied, from very simple ones (e.g., representing a proper fraction using the area model) to quite challenging ones (i.e., items targeting the dense ordering of fractions). In addition, the questionnaire included 4procedural items on standard school-taught procedures, namely on fraction operations (e.g., "compute the sum 10/63+8/9"). The students were asked to solve the conceptual tasks and explain their solutions; and to carry out the fraction operations.

The results showed no significant difference between the two age groups' overall performance, and no significant difference in the overall performance for conceptual tasks, norfor procedural tasks.

However, we found that older students relied significantly more on procedural strategies (i.e., school-taught procedures). Looking at each task separately, we found that ninth graders performed significantly better in 3 conceptual tasks, albeit relying on transformation strategies. For example, they converted fractions into decimals in order to place them on the number line; and they converted 3/2 and 13/27 into similar ones in order to compare them. Performance in procedural tasks was fairly good (at least 70% succeeded in each task), whereas performance in conceptual tasks varied widely (as expected). An interesting pattern emerged: Although the great majority of students were apt to carry out the four fraction operations, about 60% failed in tasks targeting conceptual understanding of the operations (e.g., in estimating the sum 12/13+7/8). In addition, although practically all students were able to construct an area model for a proper fraction, about 25% asserted that a shaded part of a shape that was partitioned in 3 unequal parts was "one-third"; and more than half of the students failed to represent an improper fraction, or explicitly stated that "it is not possible to take five parts out of three". Further, about half asserted that "eating 3/5 of a pie" necessarily means "eating 3 pieces of a pie" (see Task1, Figure 1). Finally, we also traced some students who were flawless in the procedural tasks, but failed in even the simplest conceptual tasks, indicating an asymmetry between the two types of knowledge. We focused on this issue in Study 2.

Study 2: Individual differences in conceptual and procedural knowledge of fractions

In Study 2 (Bempeni & Vamvakoussi, 2015) we recruited 7 ninth graders to participate in in-depth interviews. The selection of the participants was not random. First, based on their school grades, all participants could be characterized as medium to high level students in mathematics. Second, they shared the same mathematics tutor. Based on information provided by their tutor, we had reasons to expect some variation in their conceptual and procedural knowledge of fractions. We used an instrument with 30 fraction open-ended tasks, adjusting and extending the collection of conceptual tasks used in Study 1, and adding few procedural tasks (e.g., operations with mixed numbers). We added one more task focusing on the role of the unit of reference, in contexts where physical quantities were involved (see Task 2 in Table 1, albeit in multiple choice format). We also explicitly asked students not to apply typical school taught procedures in certain fraction comparison tasks that could be tackled with conceptual strategies (e.g., one proper and one improper fraction). The participants were asked to solve the tasks thinking aloud and explaining their answers.

Based on the analysis of their responses in terms of accuracy and strategy used (procedural/conceptual), the participants were distributed in three profiles. The *Conceptual-Procedural* profile consisted of three students who showed advanced conceptual knowledge of fractions, combined with procedural fluency. The *Procedural* profile consisted of three students who succeeded in practically all tasks that could be solved via the use of procedures taught at school (e.g., fraction and mixed numbers operations), but failed systematically in conceptual tasks (e.g., in representing an improper fraction; or in comparing dissimilar fractions when they were not allowed to apply the typical procedure). All three failed in both tasks targeting the role of the unit of reference. Finally, the *Conceptual* profile was represented by one student that failed in all tasks requiring procedural knowledge but managed to deal successfully with most of the conceptual tasks, including the ones targeting the role of the unit of reference.

Thus the findings of Study 2 are consistent with the findings of Hallett et al. (2010, 2012), indicating that there are individual differences in the way students combine the two types of knowledge. Moreover, they illustrated the possibility that these differences can be extreme, even at grade 9. In following studies, we attempted to further investigate this issue, shifting from qualitative to quantitative methods.

Study 3: Developing and evaluating an instrument to measure for conceptual and procedural knowledge

Bearing in mind Rittle-Johnson and Schneider’s (2015) plea for valid measures of procedural and conceptual knowledge, we developed a new instrument measuring conceptual and procedural knowledge of fractions (Bempeni, Poulopoulou, Tsiplaki, & Vamvakoussi, 2018). In its initial form, the instrument consisted of 39 items, 12 procedural items and 27 conceptual items, a total of 39 items. The procedural ones were paper-and-pencil tasks requiring knowledge of procedures taught at school (e.g., to carry out fractions operations and operations with mixed numbers, to find an equivalent fraction, to cross-multiply, to simplify complex fractions, and to compare dissimilar fractions using the standard procedure).The conceptual tasks were based on extensive literature review (e.g., Baroody & Hume, 1991; Van Hoof, Verschaffel, & Van Dooren, 2015) and on our materials from Study 1 and Study 2 which were adjusted and enriched when necessary, based on our experience with testing these tasks with students. First, we opted for multiple-choice items, instead of open ones (see also Van Hoof et al., 2015), with a view to discourage the use of procedural strategies in conceptual tasks (e.g., in comparison tasks), an issue that is highlighted in the literature (e.g., Rittle-Johnson & Schneider, 2015) and was noticeable in Study 1. Second, we added more tasks on fraction representations that proved good indicators of conceptual knowledge (or the lack thereof) in our previous studies (see, for example, Task 4 in Figure 1).

We conducted a clinical pilot study with 61 students and asked 6 mathematics education experts to assist in the evaluation of the instrument. The instrument was assessed with respect to a) face validity and content validity, through the experts’ feedback on clarity, accuracy, and relevance of the instrument; b) convergent and divergent validity, via multitrait analysis; c) internal consistency, calculating Cronbach’s alpha; and d) external consistency with the test-retest method, calculating the intra-class correlation coefficient.

The instrument showed strong face validity given that tall items were assessed as clear and accurate by the experts, who were also highly consistent with each other in rating the relevance of each item to the aim of the instrument (Content Validity Index =1>.83). All items of the procedural scale showed convergent validity and divergent validity by demonstrating high correlation with the procedural scale and low correlation with the conceptual scale, respectively. However, eight items of the conceptual scale showed low correlation with the conceptual scale or higher than expected correlation with the procedural scale. Further, the value of intra-class correlation coefficient was high (above 0.8) for all procedural items, but below 0.5 for five conceptual items.

The problematic items were removed, resulting in an instrument consisting of 14 conceptual and 12 procedural tasks, with good indicators of validity, reliability, and objectivity, which we used in

Study 4. However, removing conceptual items from the questionnaire is not an insignificant matter, and we will come back to this issue in the discussion.

1. Somebody ate $\frac{3}{5}$ of a pie. This person ate:

a. 3 pie slices **b.** 3:5 pie slices **c.** None of the previous responses

2. Maria bought one pizza from Lucullus and ate one quarter of it. John bough one pizza from Vesuvius and ate half of it. Who ate more pizza?

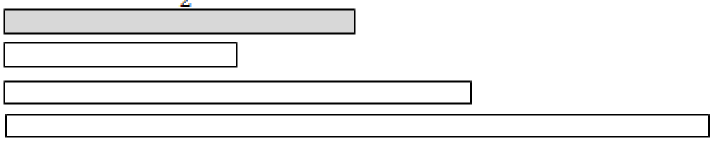
a. Mary **b.** John **c.** None of the previous responses

3. *Helen has two sock drawers. One half of the socks in the first drawer are white. One third of the socks in the second drawer are also white. How many white socks does Helen have?*

➢ Which operation do you believe is the correct one to solve this problem?

a. $\frac{1}{2}+\frac{1}{3}$ **b.** $\frac{1}{2}\cdot\frac{1}{3}$ **c.** None of the previous responses

4. The gray bar is $\frac{3}{2}$ X, where X is one of the white bars. Which one is X?



a. The first white bar **b.** The second white bar **c.** The third white bar

Figure 1: Examples of conceptual tasks

Study 4: Individual differences in conceptual and procedural knowledge of fractions: a quantitative study

In Study 4 (Bempeni et al., 2018) we administered the aforementioned instrument to 126 ninth graders and we analyzed the data using cluster analysis, testing for individual differences. Following Hallett and colleagues (2010, 2012), we used in the analysis the residualized scores of the procedural and the conceptual scales, the raw scores being the percentages of correct answers out of the total of answered questions. This is because the two scales are expected to be correlated (Rittle-Johnson & Schneider, 2015), and this method provides a way to exclude the common part of variation from both scales (Hallett et al., 2010, 2012). It should be noted that these scores do not represent absolute magnitudes of each type of knowledge; rather, they indicate discrepancies between the two types of knowledge. For example, a positive residual with respect to procedural knowledge means that a person's procedural knowledge is stronger than expected given their conceptual knowledge, and vice versa for negative residuals.

The cluster analysis used the k-means method and Euclidean distance as a distance measure and the optimal number of clusters was determined via statistical methods to be four. Two of these clusters comprised students who did not show great discrepancies between the two types of knowledge and their performance was either rather high in both type of tasks (N=22, 17.5%) or rather low in both types of tasks (N=31, 24.6%). On the contrary, the two remaining clusters comprised students who either performed better than expected in conceptual tasks given their performance in procedural tasks(*Conceptual Profile*, N=21, 16.7%); or performed better in procedural tasks given their performance in conceptual tasks(*Procedural Profile*, N=52, 41.3%).

These results supported the hypothesis that there are individual differences in the way that students combine the two type of knowledge and were consistent with the findings of our previous studies, and also with the results of Hallett et al. (2010, 2012). Moreover, our findings provided evidence that these differences may persist and remain salient for older students.

We note that within the *Procedural Profile,* we traced few extreme cases of students who excelled in the procedural items, but showed a severe lack of conceptual understanding, similar to students in Study 2. For example, one student achieved 100% score in the procedural tasks but only 14.29% in conceptual tasks, failing even in some of the simplest ones. However, we did not trace extreme cases within the *Conceptual Profile.*

Conclusions - Discussion

In a series of studies we examined secondary students' conceptual and procedural knowledge of fractions, qualitatively as well as quantitatively. The findings of these studies converge on the conclusion that there are individual differences in the way that students combine these types of knowledge (i.e., there are students who exhibit stronger procedural than conceptual knowledge, and vice versa); that these differences remain salient up to ninth grade; and that they may even be extreme (Study 2, Study 4). These findings put a challenge to the assumption that conceptual and procedural knowledge develop in a hand-over-hand manner (Rittle-Johnson & Schneider, 2015).

The findings are strengthened by the fact that we measured conceptual and procedural fraction knowledge reliably and validly via a new instrument (Study 4). However, there were several theoretical and methodological issues that we had to tackle in the process of developing the instrument, and some difficult decisions to make. First, measures of procedural knowledge were easy to construct and validate. However, this is due to the fact that we adopted a simple, perhaps over-simplified, definition of procedural knowledge (Star, 2005), as discussed in the introduction.

Second, measures of conceptual knowledge were particularly challenging. On the one hand, it is important to address various aspects of conceptual knowledge (Rittle-Johnson & Schneider, 2015), which requires the use of a great variety of tasks. On the other hand, this variety makes it difficult to construct a measure with good indicators of convergent validity, resulting to the exclusion of tasks (Study 3). Further, the use of multiple-choice items does not exclude the possibility that students in fact use procedural strategies when dealing with conceptual tasks, which was observable in Studies 1 and 2, but not in Studies 3 and 4. This might be an explanation for the fact that some conceptual tasks showed higher correlation with the procedural scale than expected (Study 3), and they also needed to be excluded. For example, locating fractions on the number line were among the problematic conceptual tasks in Study 4, presumably because the students, similarly to the students in Study 1, used transformation strategies to deal with the task. Excluding items may result in a more robust instrument; however, a lot of useful insights in students' understandings are lost. Consider, for example, the three first tasks in Figure 1. These tasks address students' understanding of the role of the unit of reference, in a context where it is necessary to take it into account (Baroody & Hume, 1991). All three had to be removed from the instrument, because the great majority of students, even the ones with overall good performance, failed. Thus, these tasks were not useful in discriminating between "conceptual" and "procedural" students but removing them meant ignoring an important aspect of conceptual knowledge that students appear to lack.

This brings us to a consistent finding across all four studies: Students fail in elementary, yet fundamental conceptual tasks. Consider Task 1 (Figure 1) which was used in all studies. This task was challenging for half the students in Study 1; for three out of seven middle-to-high level students (based on their school grades) in Study 2; and for the great majority of students who participated in Study 3. This raises certain issues for instruction. First, Study 1 indicates that conceptual knowledge does not improve from seventh to ninth grade, despite the fact that during these years Greek students recapitulate content regarding fractions and are introduced to rational numbers. However, what seems to change is that students tend to rely more on procedural strategies. This may conceal their lack of conceptual fraction understanding is some cases (e.g., when placing fractions of the number line), but not in tasks that cannot be tackled with school-taught procedures (e.g., Tasks 1-3 in Figure 1). Second, it appears that students with very poor understanding of fractions still manage to get good grades at school (Study2). Third, the majority of the students in Study 4 was placed in the *Procedural* profile (i.e., they did better in the procedural tasks, than one would expect based on their performance in the conceptual tasks). These are quite strong indications that students' school experiences are more favorable to the development of procedural knowledge. In other words, instruction appears to still over-emphasize procedural knowledge, neglecting students' conceptual difficulties with fractions (Moss & Case, 1999).

This assumption cannot, however, explain the existence of (fewer) students who have stronger conceptual than procedural knowledge. In a more general fashion, the source of individual differences in conceptual and procedural knowledge is still an open question, despite the fact that several hypotheses have been formulated. Factors such as prior knowledge in the domain (Schneider, Rittle-Johnson, & Star, 2011), cognitive profile (Gilmore & Bryant, 2008; Hallett et al., 2012), general conceptual of procedural ability as well as school attendance at different schools have been tested (Hallett et al., 2012), with unsatisfactory results. Further theorizing and research is needed in this respect.

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