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Structural properties of signed-graphic matroids

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PhD thesis

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To my family

Abstract

In this doctoral thesis we furnish structural results for signed-graphic matroids focusing mainly on two subclasses binary and quaternary signed-graphic matroids. Our purpose is to decompose the class of quaternary signed-graphic matroids and to characterize the classes of cographic signed-graphic and binary signed-graphic matroids.

We provide a characterization for the class of cographic signed-graphic matroids which is based on properties of cocircuits. To achieve this, we show that each cographic excluded-minor of signed-graphic matroids contains a Fournier triple with two non-graphic cocircuits. Furthermore, we present a characterization for binary signed-graphic matroids along with two algorithms. A polynomial algorithm which checks whether a binary matroid is isomorphic with the signed-graphic matroid of a given jointless signed graph and a recognition algorithm for binary signed-graphic matroids. Regarding tangled signed graphs, we define an operation which preserves the number of negative cycles. As a consequence, we prove that the number of negative cycles in tangled signed graphs is polynomially bounded by the number of negative cycles of signed graphs belonging to two well-defined classes.

The class of quaternary signed-graphic matroids is characterized by a decomposition theorem which states that the existence of a non-graphic and bridge-separable cocircuit which decomposes a quaternary matroid into a graphic minor and a signed-graphic minor are necessary and sufficient conditions for the matroid to be signed-graphic. As a result, we determine the building blocks of quaternary signed-graphic matroids which are graphic matroids and signed-graphic matroids which become graphic upon the deletion of any cocircuit. To this end quaternary signed-graphic matroids are studied in terms of signed graphs representing them. Moreover, we prove that hereditary properties under k-sums permit desirable graphical representations. The decomposition theorem, which is based on a new operation called star composition and k-sums, constitutes the theoretical background for a recognition algorithm.

A survey of important results of Matroid Theory as well as conjectures and recent work on the field are also presented.

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Greek summary

Τα θεωρήματα αποκλειομένων ελασσόνων και τα θεωρήματα αποσύνθεσης συμπεριλαμβάνονται στα πιο αξιόλογα αποτελέσματα της Θεωρίας Μητροειδών. Ειδικότερα, τα θεωρήματα αποσύνθεσης των μητροειδών είναι ιδιαίτερης σημασίας, επειδή οδηγούν σε πολυωνυμικούς αλγορίθμους αναγνώρισης των αντίστοιχων κλάσεων μητροειδών. Επιπλέον, τα θεωρήματα αποσύνθεσης για κλάσεις αναπαραστήσιμων μητροειδών οδηγούν σε πολυωνυμικούς αλγορίθμους για την αναγνώριση των κλάσεων των πινάκων που τα αναπαριστούν. Δυο ενδεικτικά παραδείγματα είναι το θεώρημα αποσύνθεσης των γραφικών μητροειδών του Tutte [61] και το θεώρημα αποσύνθεσης των κανονικών μητροειδών του Seymour [50]. Το θεώρημα αποσύνθεσης των γραφικών μητροειδών του Tutte οδήγησε στην υλοποίηση ενός πολυωνυμικού αλγορίθμου για την αναγνώριση των γραφικών μητροειδών και, παράλληλα, στην υλοποίηση ενός αποτελεσματικού αλγορίθμου για την αναγνώριση των πινάκων δικτύου, δηλαδή των πινάκων που αναπαριστούν γραφικά μητροειδή. Το θεώρημα αποσύνθεσης των κανονικών μητροειδών του Seymour συνέβαλε καθοριστικά στην υλοποίηση πολυωνυμικού αλγορίθμου αναγνώρισης των κανονικών μητροειδών, καθώς και στην υλοποίηση πολυωνυμικού αλγορίθμου αναγνώρισης των τ-πινάκων, οι οποίοι αναπαριστούν κανονικά μητροειδή.

Τα αποτελέσματα της Θεωρίας Μητροειδών έχουν σημαντικές εφαρμογές σε αρκετούς επιστημονικούς χώρους, όπως τα Διακριτά Μαθηματικά και η Συνδυαστική Βελτιστοποίηση. Αντιπροσωπευτικό παράδειγμα αποτελεί το θεώρημα αποσύνθεσης των κανονικών μητροειδών του Seymour [50], από το οποίο προέκυψε ένας πολυωνυμικός αλγόριθμος για την αναγνώριση των τ-πινάκων. Οι τ-πίνακες ειναι μείζονος σημασίας για τη Συνδυαστική Βελτιστοποίηση, καθώς ορίζουν μια κλάση ακέραιων προγραμμάτων, τα οποία μπορούν να επιλυθούν σε πολυωνυμικό χρόνο. Σημαντικά αποτελέσματα της Θεωρίας Μητροειδών, που αναμένεται να έχουν πολυάριθμες εφαρμογές σε πολλούς επιστημονικούς χώρους όπως τα Διακριτά Μαθηματικά και η Συνδυαστική Βελτιστοποίηση, έχουν προκύψει από το Έργο Ελασσόνων Μητροειδών, που εκπονήθηκε από τους Geelen, Gerards και Whittle [19, 20]. Το Έργο Ελασσόνων Μητροειδών, αν και δεν έχει δημοσιευθεί ακόμα στο σύνολό του, γενικεύει το Έργο των Ελασσόνων Γραφημάτων των Robertson και Seymour στα αναπαραστήσιμα μητροειδή σε πεπερασμένα πεδία. Πρόσφατα, η Θεωρία Μητροειδών συνδυάστηκε επιτυχώς με τη Θεωρία Παιγνίων στην εργασία [57] όπου ορίστηκαν παίγνια σε μητροειδή με ποικίλες εφαρμογές σε θέματα ασφάλειας.

Τα προσημασμένα-γραφικά μητροειδή προκύπτουν από τα προσημασμένα γραφήματα και αποτελούν γενίκευση των γραφικών μητροειδών που προκύπτουν από τα γραφήματα [74]. Τα προσημασμένα γραφήματα αποτελούν χρήσιμες δομές για τη μοντελοποίηση προβλημάτων και την αναπαράσταση φυσικών δικτύων, ηλεκτρικών κυκλωμάτων καώς και των αλληλεπιδράσεων που μπορεί να προκύψουν σε δομές δεδομένων και τη ροή ελέγχου σε προγράμματα υπολογιστών. Αποτελέσματα των προσημασμένων-γραφικών μητροειδών μπορούν να μεταφραστούν σε αποτελέσματα για προσημασμένα γραφήματα και, συνεπώς να συμβάλλουν στην επίλυση προβλημάτων των Διακριτών Μαθηματικών και της Συνδυαστικής Βελτιστοποίησης. Ένα μητροειδές είναι προσημασμένο-γραφικό, εάν μπορεί να αναπαρασταθεί από έναν πίνακα στο $GF(3)$ πεδίο, όπου κάθε στήλη έχει το πολύ δυο μη μηδενικά στοιχεία. Τα μητροειδή τα οποία είναι αναπαραστήσιμα σε κάθε πεδίο εκτός ίσως από το $GF(2)$, ονομάζονται περίπου-κανονικά [72]. Αναζητώντας θεωρήματα αποσύνθεσης, τα οποία οδηγούν σε αλγόριθμους αναγνώρισης για γενικότερες κλάσεις μητροειδών και κατα συνέπεια των πινάκων που τα αναπαριστούν, μελετάμε θεμελιώδεις κλάσεις μητροειδών για τις οποίες υπάρχουν ενδείξεις ότι αποτελούν τους δομικούς τους λίθους. Τα προσημασμένα-γραφικά μητροειδή έχουν προσελκύσει μεγάλο ερευνητικό ενδιαφέρον, καθώς εικάζεται ότι αποσυνθέτουν την κλάση των περίπου-κανονικών μητροειδών με τρόπο παρόμοιο με αυτόν που τα γραφικά μητροειδή αποσυνθέτουν τα κανονικά μητροειδή [33, 72].

Τα μητροειδή που αναπαρίστανται από έναν πίνακα με στοιχεία στο $GF(2)$ και $GF(4)$ πεδίο, ονομάζονται δυαδικά και τετραδικά αντίστοιχα. Ένα αντικύκλωμα ονομάζεται γραφικό αν το μητροειδές που προκύπτει μετά τη διαγραφή του είναι γραφικό, διαφορετικά ονομάζεται μη γραφικό. Έχει αποδειχθεί ότι τα δυαδικά μητροειδή [51] και τα προσημασμένα-γραφικά μητροειδή δεν είναι πολυωνυμικά αναγνωρίσιμα [33]. Ωστόσο, δεν έχει αποδειχθεί ανάλογο αποτέλεσμα για τα δυαδικά και τα τετραδικά προσημασμένα-γραφικά μητροειδή. Τα δυαδικά και τα τετραδικά προσημασμένα μητροειδή έχουν αποσυντεθεί μέσω της πράξης των κ-αθροισμάτων από τον Slilaty[55]. Επιπλέον τα δυαδικά προσημασμένα-γραφικά μητροειδή έχουν αποσυντεθεί διαγράφοντας ένα μη-γραφικό αντικύκλωμα από τους Παπαλάμπρου και Πιτσούλη [40]. Πιο συγκεκριμένα, απέδειξαν ότι ένα δυαδικό μητροειδές είναι προσημασμένο-γραφικό εάν και μόνο εάν κάποια καλά ορισμένα ελάσσονα που προκύπτουν από τη διαγραφή του αντικυκλώματος είναι γραφικά, εκτός από ένα που είναι προσημασμένο-γραφικό. Ένας από τους βασικότερους στόχους της παρούσας διδακτορικής διατριβής ήταν η γενίκευση του παραπάνω αποτελέσματος για τα τετραδικά προσημασμένα-γραφικά μητροειδή. Επιπρόσθετες επιδιώξεις ήταν ο χαρακτηρισμός της κλάσης των δυαδικών προσημασμένων-γραφικών μητροειδών, καθώς και ο χαρακτηρισμός της κλάσης των συγγραφικών προσημασμένων-γραφικών μητροειδών που διαθέτουν ένα μη γραφικό συγκύκλωμα.

Ο χαρακτηρισμός της κλάσης των δυαδικών προσημασμένων-γραφικών μητροειδών στηρίχθηκε στη μελέτη της δομή τους, αλλά και της δομής των γραφικών τους αναπαραστάσεων. Ειδικότερα, αποδείχθηκαν δομικά αποτελέσματα για τα περίπλοκα προσημασμένα γραφήματα και χαρακτηρίστηκαν γραφικά τα κυκλώματα, τα συγκυκλώματα και οι βάσεις των δυαδικών προσημασμένων-γραφικών μητροειδών. Επιπλέον δόθηκαν πίνακες αναπαράστασης για τα δυαδικά προσημασμένα-γραφικά μητροειδή και διατυπώθηκε θεώρημα που χαρακτηρίζει την κλάση τους (Θεώρημα 43). Το τελευταίο θεώρημα γενικεύει ένα γνωστό αποτέλεσμα για τα γραφικά μητροειδή το οποίο αναφέρεται στην εργασία [17]. Από το χαρακτηρισμό για την κλάση των δυαδικών προσημασμένων-γραφικών μητροειδών προέκυψαν δύο αλγόριθμοι: ένας αλγόριθμος που ελέγχει εάν ένα δυαδικό μη γραφικό μητροειδές είναι ισόμορφο με το προσημασμένο-γραφικό μητροειδές ενός προσημασμένου γραφήματος, χωρίς αρθρώσεις που δίνεται ως είσοδος (Αλγόριθμος 2), και ένας αλγόριθμος αναγνώρισης για τα δυαδικά προσημασμένα γραφικά μητροειδή (Αλγόριθμος 3). Όσον αφορά στις δομικές ιδιότητες των περίπλοκων προσημασμένων γραφημάτων προέκυψε ότι η πράξη της συστολής μιας ακμής, η οποία δεν είναι ούτε αρνητική άρθρωση ούτε χορδή αρνητικού κύκλου, διατηρεί το πλήθος των αρνητικών κύκλων. Ακόμη, αποδείχθηκε ότι το πλήθος των αρνητικών κύκλων των περίπλοκων προσημασμένων γραφημάτων είναι πολυωνυμικά φραγμένο από το πλήθος των αρνητικών κύκλων των προσημασμένων γραφημάτων που ανήκουν σε δύο καλά ορισμένες κλάσεις.

Όσον αφορά στην κλάση των συγγραφικών προσημασμένων-γραφικών μητροειδών, που διαθέτουν τουλάχιστον ένα μη γραφικό συγκύκλωμα, διατυπώθηκε ένας χαρακτηρισμός που βασίστηκε στις ιδιότητες των συγκυκλωμάτων τους (Θεώρημα 39). Το παραπάνω αποτέλεσμα γενικεύει ένα χαρακτηρισμό που δημοσίευσε ο Fournier για τα γραφικά μητροειδή [15]. Το θεώρημα που αποδείχθηκε αναφέρει ότι ένα συγγραφικό μητροειδές, το οποίο διαθέτει τουλάχιστον ένα μη γραφικό συγκύκλωμα, είναι προσημασμένο-γραφικό εάν και μόνο εάν κάθε Fournier τριάδα περιέχει το πολύ ένα μη γραφικό συγκύκλωμα. Για να προκύψει ο χαρακτηρισμός αποδείχθηκε ότι στα προσημασμένα-γραφικά μητροειδή των περίπλοκων γραφημάτων, τα οποία διαθέτουν τουλάχιστον ένα μη γραφικό συγκύκλωμα, κάθε Fournier τριάδα περιέχει το πολύ ένα μη γραφικό συγκύκλωμα. Επίσης, μελετήθηκαν τα συγγραφικά ελάσσονα των προσημασμένων-γραφικών μητροειδών ως προς τις Fournier τριάδες και δείχθηκε ότι κάθε έλασσον διαθέτει μια Fournier τριάδα που περιέχει δύο μη γραφικά συγκυκλώματα.

Για την αποσύνθεση των τετραδικών προσημασμένων-γραφικών μητροειδών, αρχικά ορίστηκαν οι κατάλληλες πράξεις. Πιο συγκεκριμένα, ορίστηκε μια νέα πράξη, η αστρική σύνθεση σε προσημασμένα γραφήματα που γενικεύει την αστρική σύνθεση που είχε οριστεί σε γραφήματα μέσω των πινάκων πρόσπτωσης [44]. Επιπλέον, μελετήθηκαν οι δομικές ιδιότητες των προσημασμένων γραφημάτων τα οποία αναπαριστούν τετραδικά προσημασμένα-γραφικά μητροειδή. Έτσι, αποδείχθηκαν δομικά αποτελέσματα για προσημασμένα γραφήματα, τα οποία είναι κυλινδρικά ή έχουν κορυφή εξισορρόπησης μετα τη διαγραφή των αρθρώσεων. Ακόμη, μελετήθηκαν τα επίπεδα προσημασμένα γραφήματα, καθώς τα κυλινδρικά προσημασμένα γραφήματα έχουν εξ' ορισμού μια επίπεδη αποτύπωση. Αποδείχθηκε ότι κάθε αρνητικός κύκλος σ' ενα προσημασμένο γράφημα Σ, όπου το προσημασμένο γράφημα χωρίς τις αρθρώσεις είναι 2-συνεκτικό, περιέχει μία αρνητική όψη. Κατά συνέπεια, κάθε αρνητικός κύκλος περιέχει ένα περιττό πλήθος αρνητικών όψεων, ενώ κάθε θετικός κύκλος περιέχει ένα άρτιο πλήθος όψεων. Τα δομικά αποτελέσματα που προέκυψαν για τα παραπάνω προσημασμένα γραφήματα, συνέβαλαν στην απόδειξη δομικών αποτελεσμάτων για την κλάση των τετραδικών προσημασμένων-γραφικών μητροειδών. Επιπρόσθετα, τα παραπάνω αποτελέσματα χρησιμοποιήθηκαν για την απόδειξη τεχνικών αποτελεσμάτων, τα οποία κρίθηκαν απαραίτητα για την αποσύνθεση των τετραδικών προσημασμένων-γραφικών μητροειδών.

Ο προσδιορισμός των δομικών ιδιοτήτων των προσημασμένων-γραφικών μητροειδών ήταν καθοριστικής σημασίας για την αποσύνθεση της κλάσης των τετραδικών προσημασμένων-γραφικών μητροειδών. Ειδικότερα, μελετήθηκαν ιδιότητες των συγκυκλωμάτων των προσημασμένων-γραφικών μητροειδών, όπως ο διαχωρισμός γεφυρών και ιδιότητες των γεφυρών των συγκυκλωμάτων, όπως η αποφυγή. Αποδείχθηκε ότι οι ιδιότητες αυτές διατηρούνται με τις πράξεις των κ-αθροισμάτων των μητροειδών, με αποτέλεσμα να προκύπτουν επιθυμητές γραφικές απεικονίσεις για τα πολυ συνδεδεμένα ελάσσονα των τετραδικών προσημασμένων-γραφικών μητροειδών. Επιπλέον, χαρακτηρίστηκαν γραφικά έννοιες, όπως τα συγκυκλώματα των προσημασμένων-γραφικών μητροειδών και αποδείχθηκε ότι κάθε προσημασμένο-γραφικό μητροειδές που έχει μη γραφικά συγκυκλώματα, διαθέτει ένα μη γραφικό συγκύκλωμα που διαχωρίζει τις γέφυρες.

Το κυριότερο νέο αποτέλεσμα της παρούσας διδακτορικής διατριβής είναι το θεώρημα αποσύνθεσης για τα τετραδικά προσημασμένα γραφικά μητροειδή (Θεώρημα 53). Το θεώρημα αυτό αναφέρει ότι ένα τετραδικό μητροειδές είναι προσημασμένο-γραφικό αν και, μόνο εάν, υπάρχει ένα μη γραφικό συγκύκλωμα το οποίο διαχωρίζει τις γέφυρες και αποσυνθέτει το μητροειδές σε ένα γραφικό έλασσον και ένα προσημασμένο-γραφικό και μη γραφικό έλασσον. Η αποσύνθεση των τετραδικών προσημασμένων-γραφικών μητροειδών πραγματοποιήθηκε με έναν συνδυασμό πράξεων, την αστρική σύνθεση και τα κ-αθροίσματα. Από την αποσύνθεση προέκυψαν οι κλάσεις των μητροειδών που αποτελούν τους δομικούς τους λίθους. Οι θεμελιώδεις κλάσεις μητροειδών που προέκυψαν από την αποσύνθεση των τετραδικών προσημασμένων-γραφικών μητροειδών είναι τα γραφικά μητροειδή και τα προσημασμένα-γραφικά μητροειδή που δε διαθέτουν μη-γραφικά συγκυκλώματα.

Τέλος θα αναφερθούν μερικές μελλοντικές προεκτάσεις της διδακτορικής μου έρευνας. Θα ήταν ιδιαίτερα ενδιαφέρον να διατυπωθεί ένας χαρακτηρισμός για τα συγγραφικά προσημασμένα-γραφικά μητροειδή, τα οποία περιέχουν ένα μη-γραφικό συγκύκλωμα που θα οδηγήσει σε έναν πολυωνυμικό αλγόριθμο αναγνώρισης της κλάση τους. Αυτός ο χαρακτηρισμός με τη σειρά του, θα οδηγήσει σ έναν πολυωνυμικό αλγόριθμο αναγνώρισης για την κλάση των δυαδικών προσημασμένων-γραφικών μητροειδών. Ένα ακόμη ανοιχτό πρόβλημα είναι η διατύπωση ενός θεωρήματος αποκλειομένων ελασσόνων για την κλάση των τετραδικών προσημασμένων-γραφικών μητροειδών, τα οποία δεν περιέχουν μη-γραφικό συγκύκλωμα, καθώς επίσης και η παρουσίαση μιας μεθόδου η οποία θα αναγνωρίζει τα μη γραφικά συγκυκλώματα που διαχωρίζουν τις γέφυρες σε ένα τετραδικό προσημασμένο-γραφικό μητροειδές.

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Chapter 1

Introduction

Matroid Theory has furnished many important results among which are excluded-minor characterizations and decomposition theorems for well-known classes of matroids. Especially decomposition theorems are of great importance since they lead to polynomial time recognition algorithms for the corresponding classes of matroids. As regards representable matroids, decomposition theorems lead also to polynomial time recognition algorithms for the associated classes of representation matrices. Two indicative examples are Tutte's decomposition theorem for the class of graphic matroids [61] and Seymour's decomposition theorem for the class of regular matroids [50]. Tutte's decomposition theorem for the class of graphic matroids not only does it lead to a polynomial time algorithm for recognizing whether a binary matroid is graphic [62], but it also leads to an efficient algorithm for recognizing the class of network matrices i.e., representation matrices of graphic matroids. Moreover, Seymour's decomposition theorem for the class of regular matroids apart from implying a polynomial time algorithm for the class of regular matroids, it leads also to a polynomial time algorithm for totally unimodular matrices.

Some of these deep results of Matroid theory have profound implications to several areas of Discrete Mathematics and to Combinatorial Optimization. A representative example is Seymour's decomposition theorem for the class of regular matroids [50] from which a polynomial time algorithm for recognizing totally unimodular matrices resulted. Totally unimodular matrices play a central role in Combinatorial Optimization since they define a class of integer programs that are solved in polynomial time. In particular, any integer program whose constraint matrix is totally unimodular can be solved as a linear program since the associated polyhedron is integral. The link between totally unimodular matrices and integral polyhedra was established by the famous theorem of Hoffman and Kruskal in [29] which states that for integral A matrices, the polyhedron $\{x|Ax \leq b, x \geq 0\}$ is integral for all integral vectors b if and only if A is totally unimodular. Further results of Matroid Theory with significant consequences are already obtained from the Matroid Minor Project by Geelen, Gerards and Whittle [19, 21]. This project, which was announced that it is completed, generalizes Robertson and Seymour's Graph Minor Project to representable matroids over finite fields. The Matroid Minor Project is expected to have numerous applications to many areas of Discrete Mathematics and Combinatorial Optimization. Moreover, Matroid Theory has been combined successfully with Graph Theory giving rise to games with various security-related applications [57].

Signed-graphic matroids are natural generalizations of graphic matroids since they arise from signed graphs [74]. Signed graphs are useful combinatorial structures which are used to model real-world problems and to represent physical networks, electrical circuits and interactions which may occur in databases or in the flow of control in a computer program. Results which are obtained for signed-graphic matroids can be translated to results for signed graphs. Thereby many problems of Discrete Mathematics and Combinatorial Optimization could be resolved with the aid of Matroid Theory. As regards representability, a matroid is signed-graphic if it can be represented by a matrix over $GF(3)$ with at most two nonzero entries in each column. Matroids which are representable over every field, except possibly $GF(2)$ are called near-regular [72]. Signed-graphic matroids constitute a class of matroids that has attracted significant research interest since it has been conjectured that they decompose the class of near-regular matroids the same way that graphic matroids decompose regular matroids. Although there are obstacles for obtaining a characterization for near-regular matroids similar to Seymour's for regular matroids, it is still hoped that a combination of operations will allow the decomposition of near-regular matroids [33].

A cocircuit is called graphic if the matroid which is obtained upon its deletion is graphic, otherwise it is called non-graphic. Given a cocircuit Y of a matroid M, Y is called bridge-separable if the elementary separators of the matroid $M\backslash Y$ can be partitioned into two classes where any two members of the same class are avoiding. Matroids which can be represented by a matrix with entries over $GF(2)$ and $GF(4)$ fields are called binary and quaternary, respectively. It has been proved that binary matroids [51] and signed-graphic matroids are not polynomially recognizable [33], however no such result exists for binary or quaternary signed-graphic matroids. Towards a characterization which leads to a recognition algorithm for binary or quaternary signed-graphic matroids, many decomposition results have been provided so far. Binary and quaterary signed-graphic matroids have been decomposed through the operation of k-sums by Slilaty in [55]. Moreover, binary signed-graphic matroids have been decomposed by deleting a non-graphic cocircuit by Papalamprou and Pitsoulis in [40]. Specifically, they proved that a binary matroid is signed-graphic if and only if some well-defined minors resulting from the deletion of a cocircuit are graphic apart from one which is signed-graphic.

The most important new result of this thesis is the decomposition theorem for quaternary signed-graphic matroids which generalizes Papalamprou and Pitsouli's decomposition theorem for binary signed-graphic matroids [40]. The decomposition theorem states that a quaternary matroid is signed-graphic if and only if there exists a non-graphic and bridge-separable cocircuit which decomposes the matroid into a signed-graphic minor and a graphic minor.

Theorem 1. *Let M be an internally 4-connected quaternary non-binary matroid with not all-graphic cocircuits. Then M is signed-graphic if and only if*

(i) there is a non-graphic cocircuit Y of M which is bridge-separable with $\mathscr{U}^+, \mathscr{U}^-$ two classes of all-avoiding bridges where $\mathscr{U}^-$ contains all the non-graphic bridges,

(ii) $M.(\bigcup_{S \in \mathscr{U}^-} S \cup Y)$ is signed-graphic and $M.(\bigcup_{S \in \mathscr{U}^+} S \cup Y)$ is graphic.

The decomposition of quaternary signed-graphic matroids is performed through a combination of operations the well-known k-sums and a new operation called star decomposition. The resulting building blocks are graphic matroids and signed-graphic matroids which become graphic upon the deletion of any cocircuit.

For a graph G, the dual matroid of the cycle matroid of G is called bond matroid or cocycle matroid of G. Any matroid which is isomorphic to the bond matroid of some graph is called cographic. The class of cographic matroids is a fundamental class of matroids since every regular matroid can be built from graphic matroids, cographic matroids and one special 10-element matroid [50]. Cographic signed-graphic matroids as well as binary signed-graphic matroids have been characterized in terms of excluded minors [45]. Nevertheless it would be desirable to obtain characterizations which would lead to recognition algorithms for the corresponding classes of matroids. To this end, we present a characterization for cographic signed-graphic matroids based on properties of cocircuits, which is motivated by Fournier's characterization for graphic matroids [15]. Moreover, we present a characterization for binary signed-graphic matroids which generalizes an analogous characterization for graphic matroids.

1.1 Organization of the thesis

The thesis is divided into three parts. The first part is introductory and includes the first three chapters. The second part includes Chapter 4 and discusses structural properties of signed graphs. The third part contains Chapters 5, 6, 7 and discusses structural properties of signed-graphic matroids focusing on the classes of binary signed-graphic and quaternary signed-graphic matroids.

In Chapter 2, we give some preliminaries which are used throughout the thesis. Moreover, we provide generalizations of graphs along with the associated incidence matrices, since they constitute representations of well-known classes of matroids.

Matroids are defined in Chapter 3, where operations and structural properties of matroids are also provided. A survey of important results for well-known classes of matroids is presented as well as recent work on the field. The focus is on representable, graphic and signed-graphic matroids which are studied in the next chapters. Several open problems and well-known conjectures are also mentioned.

Chapter 4 deals with signed graphs. We provide graphical operations which are used in the decomposition of quaternary signed-graphic matroids in Chapter 7 and structural results for signed graphs that represent quaternary signed-graphic matroids. Furthermore, we obtain structural results for the aforementioned class of matroids by determining structural properties of signed graphs representing them.

Chapter 5 is about signed-graphic matroids that is matroids which arise from signed graphs. In this chapter, we present structural results for signed-graphic matroids which are essential for the decomposition of quaternary signed-graphic matroids. Moreover, we characterize graphically matroidal notions and we investigate properties of cocircuits and their bridges such as bridge-separability and avoidance, respectively, under k-sums.

Structural properties of signed-graphic matroids which are binary or quaternary are determined in Chapters 6 and 7, respectively. Structural results for tangled signed graphs and a characterization for binary signed-graphic matroids are presented in Chapter 6. Negative cycles in tangled signed graphs and operations which preserve their number are also investigated. Furthermore a characterization for cographic signed-graphic matroids based on properties of cocircuits and two algorithms deriving from the characterization of binary signed-graphic matroids are provided.

In Chapter 7 we decompose the class of quaternary signed-graphic matroids. The main result of the thesis, the decomposition theorem for quaternary signed-graphic matroids is proved. In addition, structural results of cocircuits which are necessary for the decomposition and lead to desirable graphical representations are also presented.

Chapter 8 includes conclusions we have drawn from our research and describes our contributions. In addition, it suggests several ideas for related future research.

Chapter 2

Preliminaries

In this chapter, we define basic notions of Algebra and Graph Theory that are used throughout the thesis. Moreover, we present known results about matrices, graphs, signed graphs and biased graphs that appear in the books of Pitsoulis [44], Oxley [35], Diestel [10], Tutte [65] and in [1, 56, 61, 74]. Some more definitions will be given later at the relative chapters.

2.1 Fields, matrices and vector spaces

The set of natural numbers $\{1, 2, 3, \ldots\}$ is denoted by $\mathbb{N}$, the set of integers by $\mathbb{Z}$, the set of non-negative integers by $\mathbb{Z}_+$ and the set of reals by $\mathbb{R}$. The finite fields that appear oftenly in this thesis are $GF(2)$, $GF(3)$, $GF(4)$ and $GF(5)$. The first two of these fields are $\mathbb{Z}_2$ i.e., the field of positive integers modulo 2, and $\mathbb{Z}_3$ while the last one is $\mathbb{Z}_5$. The finite fields $GF(2)$, $GF(3)$ and $GF(4)$ are called also binary, ternary and quaternary field respectively. The two elements of the $GF(2)$ field are denoted by 0 and 1 and the operations of addition and multiplication are performed modulo 2 as follows:

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

The three elements of $GF(3)$ field are denoted by 0, 1 and -1, and the operations of addition and multiplication are performed modulo 3 as follows:

+	0	1	-1
0	0	1	-1
1	1	-1	0
-1	-1	0	1

×	0	1	-1
0	0	0	0
1	0	1	-1
-1	0	-1	1

The four elements of the $GF(4)$ field, which is not isomorphic to $\mathbb{Z}_4$, are denoted by $0, 1,\omega$ and $\omega + 1$ where $\omega^2 = \omega + 1$. The addition and multiplication tables for $GF(4)$ are as follows:

$+$	0	1	ω	$\omega+1$
0	0	1	ω	$\omega+1$
1	1	0	$\omega+1$	ω
ω	ω	$\omega+1$	0	1
$\omega+1$	$\omega+1$	ω	1	0

$\times$	0	1	ω	$\omega+1$
0	0	0	0	0
1	0	1	ω	$\omega+1$
ω	0	ω	$\omega+1$	1
$\omega+1$	0	$\omega+1$	1	ω

Any set considered throughout the thesis is finite, unless otherwise stated. For a set E, its *power-set* i.e., collection of subsets, and its cardinality will be denoted by 2^E and $|E|$, respectively. Given a tuple $(E, \mathscr{F})$, where $\mathscr{F} = (S_i : i \in I)$ is a family of subsets of E, a subset $X \subseteq E$ is *maximal* with respect to $\mathscr{F}$, if $X \in \mathscr{F}$ and there does not exist $Y \in \mathscr{F}$ such that $X \subset Y$. Furthermore, $X \subseteq E$ is *minimal* with respect to $\mathscr{F}$, if $X \in \mathscr{F}$ and there does not exist $Y \in \mathscr{F}$ such that $Y \subset X$. If X and Y are sets, then their symmetric difference $X \triangle Y$, is the set $(X - Y) \cup (Y - X)$.

A $n \times m$ matrix A with elements a_{ij} over a field $\mathbb{F}$ will be detoted by $A = (a_{ij}) \in \mathbb{F}$ and the $n \times n$ identity matrix by I_n. Given a $n \times m$ matrix $A \in \mathbb{F}$, the *column space* of A is the set of all linear combinations of the columns of A. The *rank* of a matrix A is the dimension of the column space of A. A matrix is of *full row rank* if its rank equals the number of its rows or, equivalently, if its row vectors are linearly independent. We shall write A^T for the *transpose* of a matrix A. A matrix is called *totally unimodular* if all the subdeterminants of its square submatrices are in $\{0, 1, -1\}$.

Given a field $\mathbb{F}$ and vectors $x_1, x_2, \ldots, x_m$ in $\mathbb{F}^n$ we say that the vector y is a *linear combination* of the vectors $x_1, x_2, \ldots, x_m$ if there exist scalars $a_1, a_2 \ldots, a_m \in \mathbb{F}$ such that $y = a_1x_1 + \ldots + a_mx_m$. A *linear relation* among the vectors x_i where $i \in \{1, \ldots m\}$ is an expression of the form $\sum_i a_ix_i = 0$. A set of vectors $\{x_i\}$ is said to be *linearly dependent* in $\mathbb{F}$ if there exists linear relation $\sum_i a_ix_i = 0$ such that $a_i \neq 0$ for some i. If a set of vectors is not linearly dependent, then we say that it is *linearly independent*. For brevity we shall say that the vectors x_i are linearly dependent (resp. linearly independent) when the set of vectors x_i is linearly dependent (resp. linearly independent).

2.2 Graphs

A graph $G = (V, E)$ is a pair of a finite set V and a finite set $E \subseteq V \cup V^2$. The elements of $V := V(G)$ are called vertices, while the elements of $E := E(G)$ are called edges. Each edge has a set of none, one or two vertices associated to it, which are called its end-vertices. There are four kinds of edges in a graph: (1) a *link* that has two distinct end-vertices, (2) a *loop* that has two equal end-vertices, (3) a *half-edge* with one end-vertex and (4) a *loose-edge* with no end-vertex. The set of half-edges and loops of G is denoted by J_G.

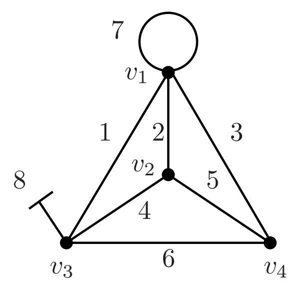


Figure 2.1: A graph

The incidence matrix of a graph G is a matrix $A_G = (a_{i,j}) \in GF(2)$ which is defined by

$$a_{i,j} = \begin{cases} 1 & \text{if non-loop edge } j \text{ is incident to vertex } v_i \\ 0 & \text{otherwise} \end{cases}$$

The full row rank matrix which is obtained from A_G by applying elementary row operations is called *full row rank incidence matrix* of G.

Example 2.2.1. *The graph in Figure 2.1 has vertex-set $V = \{v_1, v_2, v_3, v_4\}$ and edge-set $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$. The edge 7 is a loop while the edge 8 is a half-edge. The incidence matrix of the above graph is*

$$A_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Two graphs G_1 and G_2 are *isomorphic*, written $G_1 \cong G_2$, if there are bijections $\psi_1 : V(G_1) \to V(G_2)$ and $\psi_2 : E(G_1) \to E(G_2)$ such that a vertex v of G_1 is incident with an edge e of G_1 if and only if $\psi_1(v)$ is incident with $\psi_2(e)$. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $V' \subseteq V(G)$, the *induced subgraph* of V' in G is denoted by $G[V']$ and is defined by $V(G[V']) = V'$ and $E(G[V']) = \{\{v, w\} \in E(G) : v, w \in V'\}$. For $E' \subseteq E(G)$ the *induced* *subgraph* of E' in G is denoted by $G[E']$ and is defined by $E(G[E']) = E'$ and $V(G[E']) = \{v \in V(G) : v \text{ is an end-vertex of some edge in } E'\}$. If G_1 and G_2 are graphs, their *union* $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The graphs G_1 and G_2 are called *disjoint* if $V(G_1)$ and $V(G_2)$ are disjoint, then so are $E(G_1)$ and $E(G_2)$. A $v_0 - v_n$ *walk* is a subgraph of G that is defined by a sequence of vertices and edges in a consecutive manner, which starts with the vertex v_0 and ends at the vertex v_n, where $e_i = \{v_{i-1}, v_i\}$ for $i = 1, \ldots, n$. A $v - w$ walk where all vertices are distinct is called $v - w$ *path*. A *cycle* is a closed path, that is $v = w$. Any partition (V_1, V_2) of $V(G)$ for non-empty V_1 and V_2, defines an *edge cut* of G denoted by $E(V_1, V_2) \subseteq E(G)$ as the set of links incident to a vertex in V_1 and a vertex in V_2. A minimal edge cut is also called a *bond* or a *cocycle* of G. An edge cut of the form $E(v, V(G) - v)$ is called the *star* of vertex v and is denoted by $star(v)$.

Identifying two vertices u and v is the operation where u and v are replaced with a new vertex v' in $V(G)$ and $E(G)$. The *deletion of an edge* e from G results in a subgraph defined as $G\backslash\{e\} = (V(G), E(G) - \{e\})$. The *deletion of a vertex* v from G is defined as the deletion of all edges incident with v and the deletion of v from $V(G)$. The *contraction of a link* $e = \{u, v\}$ is the subgraph denoted by G/e which results from G by identifying u, v in $G\backslash e$. The *contraction of a half-edge* $e = \{v\}$ or a *loop* $e = \{v\}$ is the subgraph denoted by G/e, which results from the removal of $\{v\}$ and all half-edges and loops incident to it, while all other links incident to v become half-edges at their other end-vertex. Contraction of a loose-edge is defined as deletion. A graph G' is called a *minor* of G if it is obtained from G by a sequence of deletions and contractions of edges and deletions of vertices. For $X \subseteq E(G)$ *deletion of* X *in* G, denoted by $G\backslash X$, is the subgraph of G which is obtained by deleting all edges of X from G. Moreover, the *deletion to* X *in* G, denoted by $G|X$, is the subgraph of G consisting of the edges in X and all vertices incident to an edge in X. Equivalently $G|X$ is the graph obtained from $G\backslash(E(G) - X)$ by deleting the isolated vertices. Furthermore, the *contraction of* X *in* G, denoted by G/X is the subgraph which is obtained by contracting all edges of X while the *contraction to* X *in* G, denoted by $G.X$ is the subgraph obtained from $G/(E(G) - X)$ by deleting the isolated vertices. A graph is called *bipartite* if its set of vertices can be partitioned into two classes V_1 and V_2 such that each edge has one end-vertex in V_1 and the other in V_2. A graph on n vertices whose any two vertices are adjacent is called *complete graph* on n vertices, denoted K_n. The graph which is obtained from a 2-connected graph G by splitting a vertex $v \in V(G)$ into two vertices v_1, v_2, adding a new edge $\{v_1, v_2\}$, and distributing the edges incident to v among v_1 and v_2 such that 2-connectivity is maintained, is called an *expansion* of G at v. The operation of *twisting* (see [35, Page 148]), is defined as follows. Let G_1 and G_2 be two disjoint graphs with at least two vertices u_1, v_1 and u_2, v_2, respectively. Let G be the graph obtained from G_1 and G_2 by identifying u_1 with u_2 as the vertex $u \in V(G)$ and v_1 with v_2 as the vertex $v \in V(G)$. In *a twisting of* G *about* $\{u, v\}$ we identify, instead, u_1 with v_2 and v_1 with u_2 and we obtain a graph G' that is called a *twisted* graph of G *about* $\{u, v\}$. The subgraphs G_1 and G_2 of G and G' are called the *parts* of the twisting.

A graph is *planar* if it can be drawn in the plane without edge-crossings. Such a drawing of a planar graph is called a *planar embedding*. The continuous regions in the plane so formed by the deletion of a planar embedding of a graph are called *faces*. The face of a graph which is unbounded i.e., does not lie within a sufficiently large disc, is the *outer* face of the graph, while the other faces are the *inner* faces. It is known that the *boundary* of a face of a planar embedding of a planar graph is always a subgraph of the graph ([10] Section 4.2). Furthermore if F is a face of a planar graph G such that $G \backslash J_G$ is a 2-connected graph where J_G is the set of half-edges and loops of the graph, then the boundary of F is a cycle. We make the convention that a graph with half-edges and loops is planar if and only if the graph which is obtained after the deletion of half-edges and loops is planar.

Connectivity for graphs

There are numerous definitions of connectivity in graphs which are generalized to matroids. However, the most widespread is Tutte's definition of k*-connectivity*, since a graph and its cycle matroid have the same k-connectivity. For $k \geq 1$, a k*-separation* of a graph G is a partition $\{A, B\}$ of the edges such that $\min\{|A|, |B|\} \geq k$ and $|V(G|A) \cap V(G|B)| = k$. A graph G is called k*-connected* if there is no l-separation where $l \leq k$. A *vertical* k*-separation* of G is a k-separation $\{A, B\}$ where $V(A) \backslash V(B) \neq \emptyset$ and $V(B) \backslash V(A) \neq \emptyset$. A separation or vertical separation $\{A, B\}$ is said to be connected or to have connected parts when $G[A]$ and $G[B]$ are both connected. A *block* is defined as a maximally 2-connected subgraph of G. Loops and half-edges are blocks in a graph, since they induce a 1-separation.

2.3 Directed graphs

In a drawing of a graph without half-edges, an edge between two vertices creates a two way connection. Assigning a *direction* to an edge makes one way forward and the other backward. An edge which has been assigned a direction is called *directed edge* or *arc*. A *directed graph* $\overrightarrow{G} = (V, E)$ is a graph whose every edge is directed. The assignement of a direction to an edge is indicated by an arrow. If $e = \{v, w\} \in E(\overrightarrow{G})$ is directed from v to w then v is called the *tail* of e while w is called the *head* of e. In case $e = \{v, v\} \in E(\overrightarrow{G})$ is directed from v to v, then one occurrence of v is the *tail* of e while the other is the *head* of e. An *orientation* o of a graph $G = (V, E)$ is a function that attributes to the end-vertices of each edge $e = \{v, w\} \in E(G)$ a sign in $\{+1, -1\}$ such that $o(e, v) = -o(e, w)$. Moreover, if $e = \{v, v\} \in E(G)$, then different signs are attributed to the two occurences of v. A directed graph is obtained from any orientation of a graph as follows: for each $e = \{v, w\} \in E(\overrightarrow{G})$, v is designated as the tail of e if $o(e, v) = -1$ and w is designated as the head of e otherwise. If $e = \{v, v\} \in E(\overrightarrow{G})$, then the designation of the end-vertices is similar.

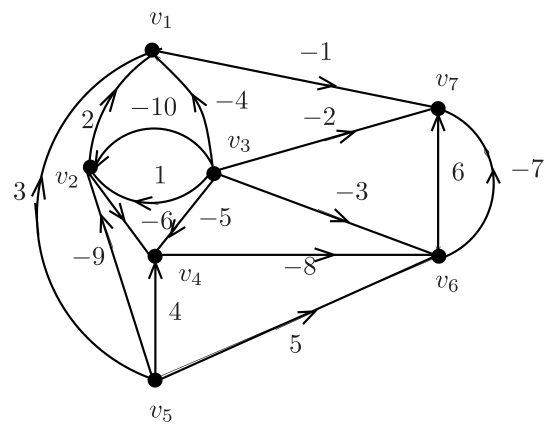


Figure 2.2: A directed graph

The incidence matrix of a directed graph $\overrightarrow{G}$ is a matrix $A_{\overrightarrow{G}} = (a_{i,j}) \in GF(3)$ defined by

$$a_{i,j} = \begin{cases} +1 & \text{if vertex } i \text{ is the head of the non-loop arc } j, \\ -1 & \text{if vertex } i \text{ is the tail of the non-loop arc } j, \\ 0 & \text{otherwise} \end{cases}$$

Example 2.3.1. *The incidence matrix $A_{\overrightarrow{G}}$ of the directed graph $\overrightarrow{G}$ in Figure 2.2 is*

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{array} \\
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\end{array}$$

The incidence matrix of a directed graph is totally unimodular. The incidence matrix of an undirected graph is totally unimodular if and only if the graph is bipartite. *Network matrices*, which are derived from directed graphs, are totally unimodular matrices and are defined as follows.

Definition 2.3.1. *Let $A = [R\ S]$ be a full row rank incidence matrix of a directed graph $\overrightarrow{G}$, where R is a basis of A. The matrix $N = R^{-1}S$ is called a network matrix.*

2.4 Signed graphs

A *signed graph* $\Sigma = (G, \sigma)$ is a graph $G = (V, E)$ together with a sign function $\sigma : E(G) \to \{+1, -1\}$ such that $\sigma(e) = -1$, if e is a half-edge and $\sigma(e) = +1$, if e is a loose-edge. The graph G is called the underlying graph of Σ. We shall denote by $V(\Sigma)$ and $E(\Sigma)$ the vertex set and edge set of a signed graph Σ, respectively. An edge e is called positive if $\sigma(e) = +1$ otherwise it is called negative. Half-edges are attributed the negative sign, while loose-edges are attributed the positive sign. In figures we use solid lines to depict positive edges and dashed edges to depict negative edges.

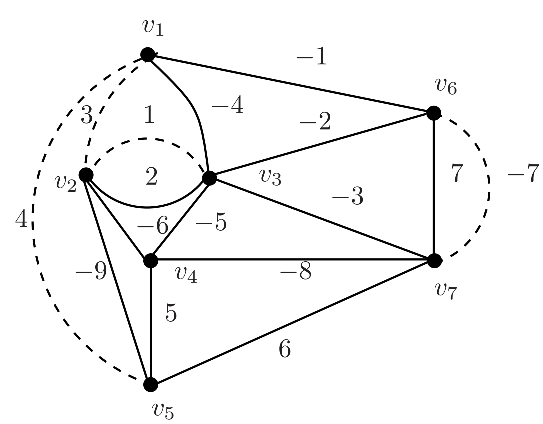


Figure 2.3: A signed graph

An *orientation* o of a signed graph $\Sigma = (G, \sigma)$ is a function that assigns to the end-vertices of each edge $e = \{v, w\} \in E(\Sigma)$ a sign in $\{+1, -1\}$ such that $-o(e, v)o(e, w) = \sigma(e)$. The *incidence* matrix of a signed graph $\Sigma = (G, \sigma)$ is a $|V(G)| \times |E(G)|$ matrix $A_\Sigma \in GF(3)$ with columns $a_e = (\alpha_{ie})_{i \in V(G)}$ for each $e \in E(\Sigma)$ defined as follows: $\alpha_{ue} = -\alpha_{ve}$ if $e = \{u, v\}$ is a positive link, $\alpha_{ue} = \alpha_{ve}$ if $e = \{u, v\}$ is a negative link, $\alpha_{ue} = 1$ or -1 if $e = \{u\}$ is a half-edge or $e = \{u, u\}$

is a negative loop, $\alpha_{ve} = 0$ if $e = \{u, u\}$ is a positive loop and $\alpha_{ie} = 0$ if $i \neq u, v$ [44].

Example 2.4.1. *The incidence matrix A_Σ of the signed graph in Figure 2.3 is*

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \end{array} \\
\begin{bmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0
\end{bmatrix}
\end{array}$$

In a signed graph, each walk $W = e_1, e_2, \ldots, e_n$ has a sign $\sigma(W) := \sigma(e_1)\sigma(e_2) \ldots \sigma(e_n)$. Therefore, a positive (resp. negative) cycle is a cycle that contains an even (resp. odd) number of negative edges. Negative loops and half-edges are considered negative cycles and are called *joints*. The set of joints of signed graph Σ is denoted by J_Σ. A signed graph without negative cycles is called *balanced*, otherwise it is called *unbalanced*. If all negative cycles of a signed graph are joints then the signed graph is called *joint unbalanced*. Following this definition, a connected component of a signed graph Σ which becomes balanced after the deletion of joints is called *joint unbalanced component* of Σ. A vertex v in an unbalanced signed graph that belongs to every negative cycle is called a *balancing vertex*. A graph G is considered to be a signed graph whose edges are all positive. Thus, signed graphs constitute a generalization of graphs. An unbalanced signed graph is called *tangled*, if it has no balancing vertex and no two vertex-disjoint negative cycles. A signed graph is *cylindrical* if it has a planar embedding with at most two negative faces.

Any operation or term on signed graphs is defined via a corresponding operation or term on the underlying graph and the sign function. In the following definitions assume that we have a signed graph $\Sigma = (G, \sigma)$. The operation of *switching* at a vertex v results in a new signed graph $(G, \bar{\sigma})$ where $\bar{\sigma}(e) = -\sigma(e)$ for each link e incident to v, while $\bar{\sigma}(e) = \sigma(e)$ for all other edges. Two signed graphs are *switching equivalent* if there exist switchings that transform the one to the other. *Deletion* of a vertex v in Σ is definedas $\Sigma \backslash v := (G \backslash v, \sigma)$. *Deletion* of an edge e in Σ is defined as $\Sigma \backslash e = (G \backslash e, \sigma)$. The *contraction* of an edge e in Σ consists of three cases: (1) if e is a positive loop, then $\Sigma / e = (G \backslash e, \sigma)$, (2) if e is a half-edge, negative loop or a positive link, then $\Sigma / e = (G / e, \sigma)$, (3) if e is a negative link, then $\Sigma / e = (G / e, \bar{\sigma})$

where $\bar{\sigma}$ is a switching at either one of the end-vertices of e. The *expansion* at a vertex v, results in a signed graph $(\bar{G}, \bar{\sigma})$, where $\bar{G}$ is the expansion of G at v, and $\bar{\sigma}$ is the same as σ except for the new edge so created by the expansion, which is given a positive sign. All remaining notions used for a signed graph are as defined for graphs (as applied to its underlying graph). For example, for some $S \subseteq E(\Sigma)$ we have that $\Sigma[S] = (G[S], \sigma)$, Σ is k-connected if and only if G is k-connected. Given a signed graph $\Sigma = (G, \sigma)$, if G is a tree, then Σ is called signed tree. *Negative* 1-*tree* of Σ is a signed tree with one more edge (link or joint) that forms a negative cycle with the signed tree. Given a connected unbalanced signed graph Σ, a negative 1-tree of Σ is denoted by T_Σ while its negative cycle is denoted by C_{T_Σ}. *Negative* 1-*path* is a connected signed graph consisting of a negative cycle and a path that has exactly one common vertex with the cycle. A signed graph such that each connected component is a negative 1-tree or a signed tree is called 1-*forest*. An 1-forest such that each connected component is a negative 1-tree, is called *negative* 1-*forest*. A *B-necklace* is a special type of 2-connected unbalanced signed graph, which is composed of maximally 2-connected balanced subgraphs Σ_i joined in a cyclic fashion as illustrated in Figure 2.4. Observe that any negative cycle in a B-necklace has to contain at least one edge from each Σ_i.

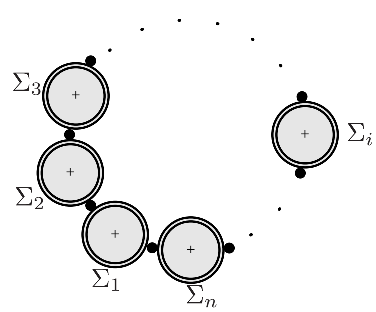


Figure 2.4: B-necklace.

Connectivity for signed graphs

For $k \geq 1$, a *k-biseparation* of a signed graph Σ is a bipartition $\{A, B\}$ of $E(\Sigma)$ such that $min\{|A|, |B|\} \geq k$ that satisfies one of the following three properties: (1) $|V(G[A]) \cap V(G[B])| = k + 1$ and both $\Sigma[A], \Sigma[B]$ are balanced (2) $|V(G[A]) \cap V(G[B])| = k$ and exactly one of $\Sigma[A], \Sigma[B]$ is balanced (3) $|V(G[A]) \cap V(G[B])| = k - 1$ and both $\Sigma[A], \Sigma[B]$ are unbalanced. A connected signed graph is called *k-biconnected* when it has no l-biseparation for $l = 0, \ldots, k - 1$. A *vertical k-biseparation* of Σ is a k-biseparation $\{A, B\}$ that has $V(A)\backslash V(B) \neq \emptyset$ and $V(B)\backslash V(A) \neq \emptyset$. A connected signed graph is called *vertically k-biconnected* when it has no vertical l-biseparation for $l = 0, \ldots, k-1$. Another definition of connectivity for signed graphs is the following. A signed graph (resp. biased graph) is k-connected if the corresponding underlying graph is k-connected.

2.5 Bidirected graphs

A *bidirected graph* $\overrightarrow{\Sigma}$ is a signed graph $\Sigma = (G, \sigma)$ with an orientation o applied to the underlying graph G such that $\sigma(e) = -o(e,u)o(e,v)$ for any edge $e = \{u, v\} \in E(\Sigma)$. Thus, positive edges have end-vertices with different signs while negative edges with the same sign. Edges with the same sign at their end-vertices are called *bidirected*, while edges with different sign at their end-vertices are called *directed*. Therefore every positive edge is a directed edge in a bidirected graph while every negative edge is bidirected. An *orientation* of a signed graph Σ is a bidirected graph, denoted by $\overrightarrow{\Sigma}$, where the positive edges of Σ become directed edges and the negative edges become bidirected. The operation of *reorienting* an edge maintains the sign of the edge while its orientation is changed according to the following. If the edge is directed, then the head of the edge becomes the tail and the tail becomes the head. Otherwise the edge is bidirected and the heads of the edge become tails and vice versa. Hence a bidirected edge (resp. directed) remains bidirected (resp. directed) when a reorientation is applied.

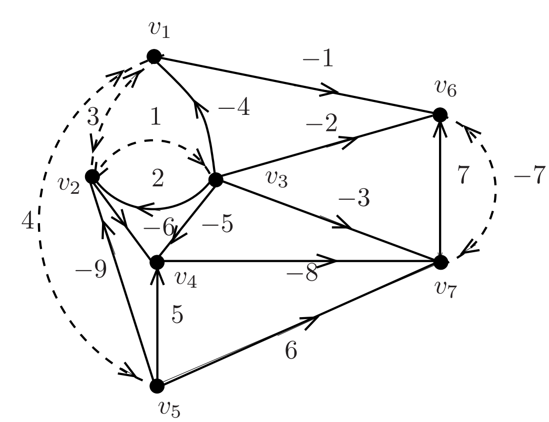


Figure 2.5: A bidirected graph

The incidence matrix of a bidirected graph $\overrightarrow{\Sigma}$, is a $|V(G)| \times |E(G)|$ matrix

$A_{\overset{\rightrightarrows}{\Sigma}} = (a_{ij}) \in \mathbb{R}$ which is defined as follows:

$$a_{ve} = \begin{cases} +1 & \text{if vertex } v \text{ is the head of the non-loop arc } e, \\ -1 & \text{if vertex } v \text{ is the tail of the non-loop arc } e, \\ +2 & \text{if vertex } v \text{ is the head of the loop arc } e, \\ -2 & \text{if vertex } v \text{ is the tail of the loop arc } e, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.5.1. *The bidirected graph which is depicted in Figure 2.5 is obtained by considering an orientation of the signed graph of Figure 2.3. The incidence matrix $A_{\overset{\rightrightarrows}{\Sigma}}$ of the bidirected graph is given below.*

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

Let $A_{\overset{\rightrightarrows}{\Sigma}} = [R|S]$ be a full row rank incidence matrix of a bidirected graph $\overset{\rightrightarrows}{\Sigma}$ where R is a basis of $A_{\overset{\rightrightarrows}{\Sigma}}$ that is a square non-singular submatrix of $A_{\overset{\rightrightarrows}{\Sigma}}$. The following algebraic definition of a *binet matrix* was given by Appa and kotnyek [1].

Definition 2.5.1. *Let $A = [R|S]$ be a full row rank incidence matrix of a bidirected graph $\overset{\rightrightarrows}{\Sigma}$, where R is a basis of A. The matrix $B = R^{-1}S$ is called a binet matrix.*

2.6 Biased graphs

A Θ-graph is the union of three internally vertex disjoint paths with the same endpoints. A *biased graph* is defined as $\Omega = (G, \Gamma)$, where Γ is a set of cycles of G satisfying the Theta Property, i.e., if C_1 and C_2 are two cycles in Γ and $G[C_1 \cup C_2]$ is a Θ-graph, then the third cycle in $G[C_1 \cup C_2]$ is also in Γ. A subgraph of Ω is called *balanced* when all cycles of the subgraph are contained in Γ, otherwise the subgraph is called unbalanced. *Bias* of a cycle of a biased graph (G, Γ) is the membership or non-membership of a cycle in Γ. The definition of two isomorphic biased graphs is the following. Two biased graphs (resp. signed graphs) are isomorphic if there is an isomorphism between their underlying graphs that preserves the bias of cycles (resp. the sign of edges). Two isomorphic biased graphs Ω_1 and Ω_2 (resp. signed graphs Σ_1 and Σ_2) are denoted by $\Omega_1 \cong \Omega_2$ (resp. $\Sigma_1 \cong \Sigma_2$).

The definition of connectivity for biased graphs that is used here is Slilaty's definition of k-biconnectivity [56]. For $k \geq 1$, a k-biseparation of a biased graph Ω is a bipartition (A_1, A_2) of the edges of Ω with $min\{|A_1|, |A_2|\} \geq k$ that also satisfies one of the following three properties:

1. $|V(A_1) \cap V(A_2)| = k + 1$ and both $\Omega[A_1]$ and $\Omega[A_2]$ are balanced
2. $|V(A_1) \cap V(A_2)| = k$ and exactly one of $\Omega[A_1]$ and $\Omega[A_2]$ is balanced
3. $|V(A_1) \cap V(A_2)| = k - 1$ and both $\Omega[A_1]$ and $\Omega[A_2]$ are unbalanced

A *vertical k-biseparation* of Ω is a k-biseparation (A_1, A_2) such that $V(A_1)\backslash V(A_2) \neq \emptyset$ and $V(A_2)\backslash V(A_1) \neq \emptyset$. A connected biased graph is called *k-biconnected* when it has no l-biseparation for $l < k$. A connected biased graph on at least k vertices is called *vertically k-biconnected* when it has no vertical l-biseparation for $l < k$.

Chapter 3

Matroid Theory

Matroids were introduced by Whitney in his article ”On the abstract properties of linear dependence” in 1935 in an attempt to axiomatize the fundamental properties of dependence that are common to graphs and matrices. Whitney presented equivalent definitions of matroids in terms of rank, independence, bases and circuits and studied matroidal properties such as connectivity, duality and representability [70]. However, the foundations of Matroid Theory were laid by van der Waerden, in 1930s in his ”Moderne Algebra”, where he captured linear and algebraic dependence axiomatically. Very few papers by Birkhoff [2], MacLane [31, 32], Dilworth [11, 12] and Rado, who worked on the combinatorial applications of matroids [46] and the representability problem [47] followed Whitney’s seminal papers on matroids. Nevertheless, since Tutte’s papers on graphs and matroids, the scientific interest in matroid theory and its applications to combinatorial theory has grown considerably.

Matroid Theory bridges several areas of Discrete Mathematics such as Algebra and Graph Theory. Not only does it provide results which lead to the resolution of important problems of Discrete mathematics and Combinatorial Optimization, but also it furnishes powerful techniques which are used for solving combinatorial optimization problems and for designing polynomial-time algorithms. Bixby and Cunningham based on Tutte’s algorithm for testing whether a binary matroid is graphic [62] provided an algorithm which converts a linear problem to a network problem or shows that no such conversion is possible [5]. Recognizing a linear problem as network problem suggests efficient solution techniques such as the network simplex method.

In this chapter, a survey of important results concerning matroids and graphs as well as recent work on the field is presented. Matroids, operations and structural properties of basic importance for the decomposition of quaternary signed-graphic matroids are also defined. Moreover, a number of open problems and well-known conjectures are mentioned.

3.1 Definitions of matroids

Matroids are also known as 'geometries' which is an abbreviation of 'combinatorial geometries'. There are thirteen different but equivalent definitions for matroids, most of them being axiomatic. The following definition of a matroid in terms of circuits axiomatizes the properties of linear dependence in vector spaces.

Definition 3.1.1. *A* matroid M *is an ordered pair* $(E, \mathscr{C})$ *of a finite set* E *and a collection* $\mathscr{C}$ *of subsets of* E *having the following three properties:*

(C_1) $\emptyset \notin \mathscr{C}$

(C_2) *If* C_1 *and* C_2 *are members of* $\mathscr{C}$ *and* $C_1 \subseteq C_2$*, then* $C_1 = C_2$

(C_3) *If* C_1 *and* C_2 *are distinct members of* $\mathscr{C}$ *and* $e \in C_1 \cap C_2$*, then there is a* $C_3 \in \mathscr{C}$ *such that* $C_3 \subseteq (C_1 \cup C_2) - e$

The set E is called the ground set of M, denoted also by $E(M)$, while $\mathscr{C}$ denotes the family of circuits of M. The independent sets of a matroid $M = (E, \mathscr{C})$ are all the subsets of E that do not contain a circuit $C \in \mathscr{C}$. A *circuit* is a minimal dependent set, that is, a dependent set all of whose proper subsets are independent. If $e \in E$ and $\{e\}$ is a circuit of M then e is a *loop* of M. If e and g are elements of M such that $\{e, g\}$ is a circuit of M, then e and g are *parallel* in M. If M has no loops and no two parallel elements, it is called a *simple matroid* denoted $si(M)$.

The following axiomatic definition of a matroid, in terms of independent sets, generalizes the properties of linear independent vectors of a vector space.

Definition 3.1.2. *Let* $\mathscr{I}$ *be a set of subsets of a set* E*. Then* $\mathscr{I}$ *is the family of independent sets of a matroid on* E *if and only if* $\mathscr{I}$ *has the following three properties:*

(I_1) $\emptyset \in \mathscr{I}$

(I_2) *If* $I \in \mathscr{I}$ *and* $I' \subseteq I$*, then* $I' \in \mathscr{I}$*.*

(I_3) *If* I_1 *and* I_2 *are in* $\mathscr{I}$ *and* $|I_1| < |I_2|$*, then there is an element* e *of* $I_2 - I_1$ *such that* $I_1 \cup e \in \mathscr{I}$*.*

A pair $(E, \mathscr{I})$ that satisfies (I_1) and (I_2) is called an *independence system*. A maximal independent set of M is called a *basis*. The set of bases of M is denoted by $\mathscr{B}(M)$ or just $\mathscr{B}$. Just as the axiom system for independent sets mirrors the properties of linear independence in vectors, the axiom system for bases is motivated by properties satisfied by the collection of bases of a (finite-dimensional) vector space.

Definition 3.1.3. *A matroid M is a pair $(E, \mathscr{B})$ in which E is a finite set and $\mathscr{B}$ is a family of subsets of E satisfying*

(B_1) $\mathscr{B} \neq \emptyset$

(B_2) *If $\mathcal{B}_1, \mathcal{B}_2 \in \mathscr{B}$ and $\mathcal{B}_1 \subseteq \mathcal{B}_2$, then $\mathcal{B}_1 = \mathcal{B}_2$.*

(B_3) *If $\mathcal{B}_1, \mathcal{B}_2 \in \mathscr{B}$ and $x \in \mathcal{B}_1 - \mathcal{B}_2$, then there is an element $y \in \mathcal{B}_2 - \mathcal{B}_1$ so that $\mathcal{B}_1 - x \cup \{y\} \in \mathscr{B}$.*

A typical example of a matroid is the *uniform matroid* $M = (E, \mathscr{B})$ with $|E| = m$ and family of bases $\mathscr{B} = \{X \subseteq E : |X| = n\}$ for two non-negative integers m and n with $n \leq m$. The latter matroid is denoted by $U_{n,m}$. If T is a spanning tree in a connected graph G and $f \in E(G) - E(T)$, then $T \cup f$ contains a unique cycle which is called the *fundamental cycle of f with respect to T*. The following result is a straighforward generalization of this.

Proposition 1. *If $\mathcal{B}$ is a basis of a matroid $M = (E, \mathscr{I})$ and $f \in E(M) - \mathcal{B}$, then there exists a unique circuit $C(\mathcal{B}, f)$ that is contained in $\mathcal{B} \cup \{f\}$. Moreover, $C(\mathcal{B}, f)$ contains f.*

The circuit $C(\mathcal{B}, f)$ of M is called the *fundamental circuit of f with respect to basis* $\mathcal{B}$. Moreover, for each $f \in E(M) - \mathcal{B}$ we define $P_f \subseteq \mathcal{B}$ such that $P_f \cup \{f\}$ is the unique circuit contained in $\mathcal{B} \cup \{f\}$, that is $C(\mathcal{B}, f) = P_f \cup \{f\}$. Two basic notions in Linear Algebra is that of the dimension of a vector space and the span of a set of vectors. The dimension of a vector space, which is the number of vectors in any of its basis, is generalized to matroids as follows.

Definition 3.1.4. *Let E be a finite set. A function $r : 2^E \to \mathbb{N} \cup \{0\}$ is the rank function of a matroid on E if and only if the following are satisfied for all $X, Y \subseteq E$:*

(R_1) $0 \leq r(X) \leq |X|$

(R_2) *If $X \subseteq Y$ then $r(X) \leq r(Y)$*

(R_3) *If $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$*

In a vector space V, a vector x is in the span of $\{x_1, x_2, \ldots, x_n\}$ if the subspaces spanned by $\{x_1, x_2, \ldots, x_n\}$ and $\{x_1, x_2, \ldots, x_n, x\}$ have the same dimension. The span of a set of vectors is generalized to matroids as follows.

Definition 3.1.5. *Given a matroid $M = (E, r)$ the closure operator is a set function $cl : 2^E \to 2^E$ defined as*

$$cl(X) = \{y \in E : r(X \cup \{y\}) = r(X)\}$$

for any $X \subseteq E$.

A set and its closure have the same rank as shown in the next lemma.

Lemma 3.1.1. *For every subset X of the ground set of a matroid M,*

$$r(X) = r(cl(X)).$$

The following result characterizes matroids in terms of their closure operators.

Theorem 2. *A function $cl : 2^E \to 2^E$ is the closure operator of a matroid $M = (E, \mathscr{I})$ if and only if the following are satisfied for all $X, Y \subseteq E$:*

(CL_1) *If $X \subseteq E$ then $X \subseteq cl(X)$*

(CL_2) *If $X \subseteq Y \subseteq E$ then $cl(X) \subseteq cl(Y)$*

(CL_3) *If $X \subseteq E$ then $cl(cl(X)) = cl(X)$*

(CL_4) *If $X \subseteq E$, $x \in E$, $y \in cl(X \cup \{x\}) - cl(X)$ then $x \in cl(X \cup \{y\})$.*

What is remarkable about Matroid Theory is the existence of a theory of duality. This theory generalizes the concepts of orthogonality in vector spaces and planarity in graphs.

Theorem 3. *Let M be a matroid and $\mathscr{B}^*(M) = \{E(M) - \mathcal{B} : \mathcal{B} \in \mathscr{B}(M)\}$. Then $\mathscr{B}^*(M)$ is the set of bases of a matroid on $E(M)$.*

The matroid whose ground set is $E(M)$ and whose set of bases is $\mathscr{B}^*(M)$ is called the *dual matroid* of M and is denoted by M^*. For any matroid M, it holds that $(M^*)^* = M$ and $r(M) + r^*(M) = |E(M)|$. The prefix 'co' is used when we refer to the dual notion of a matroid and an asterisk is used for the corresponding notation. Thereby circuits, independent sets and flats of M^* are called cocircuits, coindependent sets and coflats of M. Furthermore $\mathcal{I}^*(M)$ is the family of coindependent sets of M, $\mathscr{B}^*(M)$ is the family of cobases of M, $\mathscr{C}^*(M)$ is the family of cocircuits of M, $r^*(M)$ is the corank function of M e.t.c.

An attractive feature about matroids is that they have an algorithmic definition. More precisely they can be defined as the output of an algorithm to the following optimization problem on independence systems.

Problem 3.1.1 (Bixby 1981)**.** *Let $(E, \mathscr{I})$ be an independence system and let w be a weight function from E into $\mathbb{R}$. Define the weight $w(X)$ of any non-empty subset X of E by*

$$w(X) = \textstyle\sum_{x \in X} w(x)$$

and $w(\emptyset) = 0$. The maximization problem for $(E, \mathscr{I})$ is to maximize $w(X)$ such that X is a maximal member of $\mathscr{I}$.

Any set $X \in \mathscr{B}$, where $\mathscr{B}$ is the collection of bases of $(E, \mathscr{I})$, is a feasible solution to the above optimization problem, while the set $X \in \mathscr{B}$ that maximizes $w(X)$ is called an *optimum* solution.

Perhaps the most intuitive algorithm for solving Problem 3.1.1 is Greedy Algorithm which is described in Algorithm 1 [44]. The algorithmic definition for matroids which was based on Greedy Algorithm was provided by Jack Edmonds [13, 14]. In this way he established the natural connection of matroids with combinatorial optimization.

Algorithm 1: GREEDY

Input: independence system $(E, \mathscr{I})$, function $w : E \to \mathbb{R}$
Output: set $X \in \mathscr{B}$
Sort E such that $w(e_1) \geq w(e_2) \geq \ldots \geq w(e_{|E|})$
$X := \emptyset$
for $i = 1, \ldots, |E|$ **do**
  if $X \cup \{e_i\} \in \mathscr{I}$ **then**
    $X := X \cup \{e_i\}$
  end if
end for
return X

Theorem 4. *An independence system $(E, \mathscr{I})$ is a matroid if and only if the Greedy algorithm has an optimum solution for the maximization problem 3.1.1.*

Two matroids M_1 and M_2 are *isomorphic* and we write $M_1 \cong M_2$, if there is a bijection $\phi : E(M_1) \to E(M_2)$ such that $X \in \mathscr{C}(M_1)$ if and only if $\phi(X) \in \mathscr{C}(M_2)$.

3.2 Operations and structural properties of matroids

Many important results of Matroid Theory are linked with the concept of decomposition. Operations and structural properties of matroids which are used for the decomposition of well-known classes of matroids such as regular, binary signed-graphic and quaternary signed-graphic matroids are presented in the following.

3.2.1 Deletion and contraction

The operations of deletion and contraction for signed graphs are generalized to corresponding operations for matroids. Moreover, these operations are dual to each other.

Proposition 2. *For a matroid $M = (E, \mathscr{C})$ and $X \subseteq E$, the set*

$$\mathscr{C}(M \backslash X) = \{C \subseteq E - X : C \in \mathscr{C}(M)\},$$

is the family of circuits of a matroid on $E - X$ which is denoted by $M \backslash X$.

The matroid $M \backslash X$ is called *the deletion of X from M*.

Proposition 3. *For a matroid $M = (E, \mathscr{C})$ and $X \subseteq E$, the set*

$$\mathscr{C}(M|X) = \{C \subseteq X : C \in \mathscr{C}(M)\},$$

is the family of circuits of a matroid on X which is denoted by $M|X$.

The matroid $M|X$ is called the *deletion to X in M* or the *restriction of M to X*. Moreover, the matroid $M|X$ is equal to the deletion of $E - X$ from M that is $M|X = M \backslash E - X$.

Proposition 4. *For a matroid $M = (E, \mathscr{C})$ and $X \subseteq E$, the set*

$$\mathscr{C}(M/X) = \textit{minimal nonempty}\{C - X : C \in \mathscr{C}(M)\},$$

is the family of circuits of a matroid on $E - X$ which is denoted by M/X.

The matroid M/X is called *the contraction of X from M*. Alternatively, the contraction of X from M is defined as $M/X = (M^* \backslash X)^*$.

Proposition 5. *For a matroid $M = (E, \mathscr{C})$ and $X \subseteq E$, the set*

$$\mathscr{C}(M.X) = \textit{minimal nonempty}\{C \cap X : C \in \mathscr{C}(M)\},$$

is the family of circuits of a matroid on X which is denoted by $M.X$.

The matroid $M.X$ is called the *contraction to X in M*. Alternatively, the contraction to X in M is defined as $M.X = (M^*/X)^*$.

Proposition 6. *For a matroid M and $X, Y \subseteq E(M)$ it holds*

(*i*) $(M \backslash X)^* = M^*/X$

(*ii*) $(M/X)^* = M^* \backslash X$

(*iii*) $(M|X)|Y = M|Y$

For a matroid M and disjoint $X, Y \subseteq E(M)$ the matroid $M\backslash X/Y$ is called a *minor* of M. If X or Y are nonempty then it is called *proper minor*. A class of matroids $\mathscr{M}$ is called *minor-closed* if every minor of a matroid $M \in \mathscr{M}$ is also a member of the class.

Proposition 7. *For a matroid M and X, Y disjoint subsets of $E(M)$ we have*

(*i*) $(M\backslash X)\backslash Y = M\backslash(X \cup Y)$

(*ii*) $(M/X)/Y = M/(X \cup Y)$

(*iii*) $(M/X)\backslash Y = (M\backslash Y)/X$

3.2.2 Connectivity for matroids

A structural property of basic importance for many problems of matroid theory, such as decomposition characterizations for classes of matroids, is connectivity. There are numerous definitions of connectivity for matroids. The one employed here is Tutte's definition of n-connectivity for matroids, which generalizes the definition of n-connectivity for graphs.

For a matroid M and a positive integer k, a partition (A, B) of $E(M)$ is a *k-separation* of M if $min\{|A|, |B|\} \geq k$ and $r(A) + r(B) \leq r(M) + k - 1$. A matroid M is *k-connected* if it has no l-separation for any $1 \leq l < k$. A *vertical k-separation* (A, B) is a k-separation for which $V(A)\backslash V(B) \neq \emptyset$ and $V(B)\backslash V(A) \neq \emptyset$. A k-separation (A, B) is called *exact* when $r(A) + r(B) = r(M) + k - 1$. If (A, B) is a k-separation of a signed-graphic matroid $M(\Sigma)$ such that $\Sigma[A], \Sigma[B]$ are connected, then (A, B) is called *connected k-separation* or *k-separation with connected parts.* A k-separation (A, B) of a matroid M is *minimal* if $min\{|A|, |B|\} = k$. For $k \geq 2$ a k-connected matroid is called internally $(k+1)$-connected if it has no non-minimal k-separations.

There are many definitions of a separator of a matroid, one of them is the following.

Definition 3.2.1. *For a matroid $M = (E, \mathscr{C})$ a set $X \subseteq E$ is called a* separator *of M if any circuit $C \in \mathscr{C}$ is contained in either X or $E - X$.*

It follows from Definition 3.2.1 that both E and $\emptyset$ are trivial separators for any matroid. Minimal nonempty separators will be called *elementary separators*.

The separators of a matroid are characterized by the property that the rank of the matroid equals the rank of a separator and the rank of the rest of the elements.

Proposition 8. *For a matroid $M = (E, \mathscr{C})$ some set $X \subseteq E$ is a separator of M if and only if $r(X) + r(E - X) = r(E)$.*

The following corollary characterizes the separators of a matroid with respect to the operations of deletion and contraction.

Corollary 1. *Given a matroid M, a set $X \subseteq E(M)$ is a separator of M if and only if $M \backslash X = M/X$.*

A matroid M and its dual M^* have the same separators as it is stated in the following corollary.

Corollary 2. *Given a matroid M, a set $X \subseteq E(M)$ is a separator of M if and only if X is a separator of M^*.*

Define a relation γ on the ground set $E(M)$ of a matroid M by $e\gamma f$ if either $e = f$, or M has a circuit containing $\{e, f\}$.

Proposition 9 (Oxley [35] Proposition 4.1.2)**.** *For every matroid M, the relation γ is an equivalence relation on $E(M)$.*

The equivalence classes defined by γ are called the *(connected) components* of M. Hence every loop and every coloop is a component of M. Until now it was sufficient to focus on 3-connected matroids, however, it seems that 3-connectivity is no longer enough. On the other hand, the notion of 4-connectivity is too strong since, for example, neither matroids of complete graphs nor projective spaces are 4-connected. Many definitions of connectivity intermediate between 3-connectivity and 4-connectivity have been given so far. Three of these connectivities are presented in the following: the first one is vertical 4-connectivity, a minimal weakening of 4-connectivity that allows projective spaces to be 4-connected. Another type is *sequential 4-connectivity* which was presented by Geelen and Whittle and allowed them to find an analogue of Tutte's Wheels and Whirls Theorem for sequential 4-connected matroids [23]. A third type is *fork connectivity* that is a weakening of 4-connectivity related to a generalization of Δ-Y exchange, which was introduced by Oxley et al. [26].

3.2.3 Avoidance and bridge-separability

Given a cocircuit Y of a matroid M, the elementary separators of the matroid $M\backslash Y$ are called the *bridges* of Y in M. If B is a bridge of Y in M then the matroid $M.(B \cup Y)$ is called *Y-component* of M. Moreover, $\pi(M, B, Y)$ is a partition of Y such that two elements are in the same set if they are in exactly the same cocircuits of $M.(B \cup Y)$. Equivalently, two elements are in the same set of $\pi(M, B, Y)$ if and only if one of them is a loop, or if the two elements are parallel to each other in the matroid $M.(B \cup Y)$. If M is a binary matroid then it holds that $\pi(M, B, Y) = \mathcal{C}^*(M.(B \cup Y)|Y)$ [63]. The latter equation is of central importance for the decomposition of quaternary signed-graphic matroids, since it is used in many proofs of the results to follow. The following two definitions appear in [40] and [63].

Definition 3.2.2. *Two bridges B_1 and B_2 of a cocircuit Y in a matroid M are avoiding if there exist $S_1 \in \pi(M, B_1, Y)$ and $S_2 \in \pi(M, B_2, Y)$ such that $S_1 \cup S_2 = Y$.*

Definition 3.2.3. *A cocircuit Y is called bridge-separable if its bridges can be partitioned into two classes such that all members of the same class are pairwise avoiding.*

Two bridges B_1, B_2 which are not avoiding are called *overlapping*, or equivalently, we say that B_1 overlaps B_2. If the matroid $M\backslash Y$ has more than one bridge, then Y is called *separating* cocircuit, otherwise it is called *non-separating*. If U is a set of bridges of a cocircuit Y of a matroid M such that each pair of bridges in U is avoiding, then we shall say that U has *all-avoiding bridges*. Consequently, if all the bridges of Y are pairwise avoiding, then we say that Y has *all-avoiding bridges*. Given two classes of bridges $\mathscr{U}^+$ and $\mathscr{U}^-$ of a cocircuit Y of a matroid M, the sets U^+ and U^- denote the union of the bridges in the classes $\mathscr{U}^+$ and $\mathscr{U}^-$ respectively.

3.2.4 k-sums for matroids

For reason of completeness we quote the definitions of generalized parallel connection, of modular sums and of k-sums $k \in \{1, 2, 3\}$ for matroids. The definition of k-sum $k \in \{1, 2\}$ for matroids appears in [50], while the 3-sum of matroids is defined in [55]. Before these definitions, a modular set is characterized in terms of independent sets in the following proposition.

Proposition 10 (Brylawski [7] Proposition 3.18)**.** *Let $M = (E, \mathscr{I})$ be a matroid, then a set S is modular if and only if for all independent subsets $I \subseteq E(M) - S$, $I \cup \{p\}$ is independent for all $p \in S$ if and only if $I \cup I'$ is independent for all independent subsets I' of S.*

If M is a matroid, then a *line* L is a flat of rank 2. A positive triangle in a signed graph Σ and a subgraph $\Sigma_{2,4}$ of Σ (see Figure 3.1) are modular lines in $M(\Sigma)$.

Given two matroids $M_1 = (E_1, r_1)$ and $M_2 = (E_2, r_2)$ having a common restriction N ($M_1|T = M_2|T = N$ where $E_1 \cap E_2 = T$), it is natural to seek for a way to stick these matroids together along N. There are examples in the litterature which show that this may not be always possible (see Example 7.2.4 [35]), however, an operation which permits this is generalized parallel connection [7, 35]. Let M_1 and M_2 be matroids with ground sets E_1 and E_2 such that $E_1 \cap E_2 = T$ and $M_1|T = M_2|T$. When $si(M_1|T)$ is a modular flat of $si(M_1)$, the *generalized parallel connection*, $P_T(M_1, M_2)$, is the matroid on $E_1 \cup E_2$ whose flats are those subsets X of $E_1 \cup E_2$ such that $X \cap E_1$ is a flat of M_1 and $X \cap E_2$ is a flat of M_2. The *modular sum* of M_1 and M_2 along a flat T that is modular in at least one of M_1 or M_2 is the matroid $P_T(M_1, M_2)\backslash T$.

1-sum

If M_1 and M_2 are matroids on two disjoint sets E_1 and E_2 respectively, we define M to be the matroid on $E_1 \cup E_2$ in which a set is a circuit if and only if it is a circuit of one of M_1, M_2. If E_1 and E_2 are not empty, then M is called the 1-sum of M_1 and M_2, denoted by $M_1 \oplus M_2$.

Lemma 3.2.1 (Seymour [50])**.** *If M is the 1-sum of M_1 and M_2 then (E_1, E_2) is a 1-separation of M. Conversely, if (E_1, E_2) is a 1-separation of M, then M is the 1-sum of $M|E_1$ and $M|E_2$.*

2-sum

Let M_1, M_2 be two matroids on two sets E_1 and E_2 with $E_1 \cap E_2 = \{z\}$, where z is not a loop or coloop of either M_1 or M_2 and with $|E_1|, |E_2| \geq 3$. Let M be the matroid on $E = (E_1 \cup E_2) - \{z\}$, in which $X \subset E$ is a circuit if and only if either

(i) X is a circuit of one of M_1, M_2 or

(ii) $(X \cap E_i) \cup \{z\}$ is a circuit of M_i $(i = 1, 2)$.

M is indeed a matroid, called the 2-sum of M_1 and M_2.

Lemma 3.2.2 (Seymour [50]). *If M is the 2-sum of M_1 and M_2 then $(E_1 - E_2, E_2 - E_1)$ is a 2-separation of M. Conversely, if (E_1, E_2) is a 2-separation of M of order 2 and z is a new element, then there are matroids M_1 and M_2 on $E_1 \cup \{z\}, E_2 \cup \{z\}$ respectively such that M is the 2-sum of M_1 and M_2.*

3-sum

Given two matroids M_1 and M_2 both containing a line L that is modular in at least one of M_1 and M_2, we define the 3-sum $M_1 \oplus_3 M_2$ as the modular sum of M_1 and M_2 along L.

3.3 Representable Matroids

Ever since matroids were introduced by Whitney, representable matroids have attracted significant research interest. Lately due to Rota's conjecture and the Matroid-Minors Project of Geelen, Gerard and Whittle [21], matroid representation theory has been one of the most active areas of research in the field.

3.3.1 Matroids arise from matrices

Matrices give rise to a fundamental class of matroids, the class of representable matroids. Specifically the columns of a $m \times n$ matrix A with entries in a field $\mathbb{F}$ are elements of a matroid while the minimal linearly dependent sets of columns of A constitute the family of circuits of the matroid.

Theorem 5. *Let E be a finite set of vectors from a vector space over some field $\mathbb{F}$ and let $\mathscr{C}$ be the collection of all minimally linearly dependent subsets of E; then $M = (E, \mathscr{C})$ is a matroid called vector matroid.*

Given a matrix A over the field $\mathbb{F}$, the vector matroid of A is denoted by $M[A]$. Any matroid isomorphic to the vector matroid $M[A]$ is called an $\mathbb{F}$-representable matroid. The dual of an $\mathbb{F}$-representable matroid M is $\mathbb{F}$-representable. Matroids that are representable over the finite fields $GF(2)$, $GF(3)$ and $GF(4)$ are called binary, ternary and quaternary respectively. *Regular* are called the matroids which are representable over every field. Equivalently, a matroid is called regular if it can be represented by a totally unimodular matrix.

Example 3.3.1. *Let A be the following matrix over the field $\mathbb{R}$ of real numbers.*

$$A = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

$M[A]$ has ground set $E = \{1,2,3,4,5,6,7\}$ and 14 circuits some of which are $\{2,3,4\}, \{1,3,5\}$, $\{1,2,6\}, \{1,2,3,7\}$, $\{5,2,3,6\}, \{5,2,7\}$. The linear independent sets of vectors $\{1\}, \{2\}$, $\{2,3\}, \{1,2\}$, $\{5,2\}, \{2,7\}$ are independent sets of the vector matroid of A.

Let A be a $m \times n$ matrix over a field $\mathbb{F}$, then the ground set of the vector matroid $M[A]$ is the set E of column labels of A. In general, the matroid $M[A]$ does not determine the matrix A uniquely. Operations which leave the matroid $M[A]$ unchanged when performed on A are defined in the following [35].

1. Interchange two rows.
2. Multiply a row by a non-zero member of $\mathbb{F}$.
3. Replace a row by the sum of that row and another.
4. Adjoin or remove a zero row.
5. Interchange two columns (the labels moving with the columns).
6. Multiply a column by a non-zero mamber of $\mathbb{F}$.
7. Replace each matrix entry by its image under some automorphism of $\mathbb{F}$.

$$\begin{array}{cc} & \begin{array}{cc} e_1 \;\; e_2 \;\; \ldots e_r & e_{r+1} \;\; \ldots \;\; e_n \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ \vdots \\ e_{r-1} \\ e_r \end{array} & \left[\begin{array}{c|c} \quad I_r \quad & \quad D \quad \end{array}\right] \end{array} \qquad D = \begin{array}{cc} & \begin{array}{cccc} e_{r+1} & e_{r+2} & \ldots & e_n \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ \vdots \\ e_{r-1} \\ e_r \end{array} & \left[\begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{array}\right] \end{array}$$

The first three operations are called *elementary row operations*. The seventh operation differs from the first six in that it is based on a property of a field $\mathbb{F}$. By a sequence of operations of (1)-(5) the matrix A can be reduced to a matrix $[I_r|D]$ where I_r is the $r \times r$ identity matrix and D is a $r \times (n-r)$ matrix over

$\mathbb{F}$. Let us suppose that $e_1, e_2, \ldots, e_n$ are the column labels of the matrix $[I_r|D]$, then $e_1, e_2, \ldots, e_r$ are the elements of a basis for $M[A]$. We shall always assume that $e_1, e_2, \ldots, e_r$ are the labels, in order, of the rows of D. Moreover, it is natural to label the rows of D in order, by $e_1, e_2, \ldots, e_n$. The matrices $[I_r|D]$ and D are representation matrices for $M[A]$ over $\mathbb{F}$ and are called *standard representation matrix* and *compact representation matrix* for $M[A]$, respectively. Two representations A_1 and A_2 of a matroid are *equivalent* if and only if A_1 can be transformed to A_2 via a sequence of the operations (1)-(7). A matroid is *uniquely representable* over a field $\mathbb{F}$ if all of its $\mathbb{F}$-representations are equivalent. Binary matroids are uniquely $GF(2)$-representable, ternary matroids are uniquely $GF(3)$-representable and 3-connected quaternary matroids are uniquely representable over $GF(4)$ [30]. On the other hand, a quaternary matroid that is not 3-connected has arbitrarily many inequivalent representations.

In 1988 Kahn conjectured that 3-connectivity was sufficient to pose a bound on the number of inequivalent representations. Oxley, Vertigan and Whittle verified Kahn's Conjecture for $q = 5$ by showing that a 3-connected matroid has at most six inequivalent $GF(5)$-representations [36]. Moreover, they showed that there are no bounds in the number of inequivalent representations of a 3-connected matroid over a finite field $GF(q)$ with at least seven elements [36]. Specifically they gave counterexamples using free spikes and free swirls and therefore they disproved Kahn's conjecture. On the contrary Geelen and Whittle proved that, when q is prime, the number of inequivalent $GF(q)$-representations of 4-connected matroids is bounded [24]. Excluding $U_{3,6}$ as a minor, Geelen et al. [22] proved that Kahn's Conjecture holds for the class of matroids representable over a fixed finite field with no $U_{3,6}$-minor. Tipless free 3-spike and the rank-3 free swirl are matroids isomorphic to $U_{3,6}$.

3.3.2 Characterizations and decomposition theorems

Excluded minor characterizations over a field

The ideal method of characterizing a class of $\mathbb{F}$-representable matroids for some finite field $\mathbb{F}$ is to provide a polynomial time algorithm determining whether a matroid given by, say, an independence oracle is $\mathbb{F}$-representable. Unfortunately, Seymour has proved that there is no polynomial time algorithm for testing if a matroid given by an independence oracle is binary [51]. Moreover, this result extends to all $GF(q)$-representable matroids for all prime powers q. Consequently, we can only hope for theorems that prove in polynomial time that a given matroid is *not* representable over a field $\mathbb{F}$. Such theorems are those that characterize a class of

$\mathbb{F}$-representable matroids by listing its *excluded minors*, that is matroids that are minor minimal with respect to not being in the class. These characterizations constitute a *short proof* of non-representability since they provide a method for proving non-representability for a finite field that requires just a few rank evaluations.

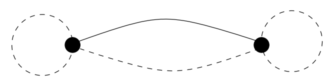


Figure 3.1: The signed graph $\Sigma_{2,4}$ such that $M(\Sigma_{2,4}) = U_{2,4}$

Excluded minor characterizations have been proved for $GF(2)$, $GF(3)$ and $GF(4)$ fields. In 1958, Tutte found the unique excluded minor of $GF(2)$ representability.

Theorem 6 (Tutte [59, 60])**.** *A matroid is binary if and only if it has no minor isomorphic to $U_{2,4}$.*

Hence Tutte gave a short proof of non-representability over $GF(2)$ since we can verify that a given minor is isomorphic to $U_{2,4}$ in a constant number of rank evaluations. In 1979, Bixby (attributing the result to Reid [4]) and Seymour [49] published independently a proof of the excluded minors of $GF(3)$-representable matroids using different techniques.

Theorem 7. *A matroid is ternary if and only if it has no minor isomorphic to $U_{2,4}, U_{3,5}, F_7$ and F_7^*.*

In 1970, based on Tutte's result for $GF(2)$ and Reid's unpublished result for $GF(3)$, Rota conjectured that for any finite field $\mathbb{F}$, the class of $\mathbb{F}$-representable matroids has only finitely many excluded minors. The latter conjecture is known as Rota's Conjecture. In 2000 Geelen, Gerards and Kapoor gave the list of the excluded minors of quaternary matroids [18].

Theorem 8. *A matroid is quaternary if and only if it has no minor isomorphic to $U_{2,6}, U_{4,6}, P_6,\ F_7^-, (F_7^-)^*, P_8$ and $P_8^{=}$.*

The fields for which Rota's Conjecture is known to be true are those that guarantee uniqueness of the representations of matroids. For finite fields larger that $GF(4)$ there are many obstacles in proving Rota's Conjecture. For this reason, current research is concentrated on developing techniques that would lead to a general resolution of Rota's Conjecture. A weakening of the latter conjecture, i.e., there is a short proof of non-representability over any finite field $\mathbb{F}$, seems more approachable for larger fields than $GF(4)$. This weaker conjecture has been resolved for $GF(5)$ by Geelen et al. [22].

Algebraic characterizations

The problem of characterizing a class of $\mathbb{F}$-representable matroids for some finite field $\mathbb{F}$ was extended to an analogous problem for sets of fields. The first result, an algebraic characterization for the class of regular matroids, was proved by Tutte. More precisely, Tutte proved that if a matroid M is representable over $GF(2)$ and M is representable over a second field of characteristic other than two, then M is regular. Moreover he showed that a matroid is regular if and only if it is representable over all fields. According to Tutte's algebraic characterization for regular matroids, there are exactly two possibilities for a class of matroids $\mathscr{M}$ representable over all fields in a set of fields $\mathcal{F}$ containing $GF(2)$. Either all fields have characteristic 2 and $\mathscr{M}$ is the class of binary matroids, or $\mathscr{M}$ is the class of regular matroids. Five equivalent characterizations for binary matroids are stated in the next theorem.

Theorem 9. *The following statements are equivalent for a matroid M:*

(*i*) *M is binary*

(*ii*) *If C is a circuit and C^* is a cocircuit, then $|C \cap C^*|$ is even*

(*iii*) *If C_1 and C_2 are circuits, then $C_1 \triangle C_2$ is a disjoint union of circuits.*

(*iv*) *The symmetric difference of any set of circuits is a disjoint union of circuits.*

(*v*) *If $\mathcal{B}$ is a basis and C is a circuit, then $C = \triangle_{e \in C-B} C(\mathcal{B}, e)$.*

The problem of characterizing a class of matroids that are representable over all fields in a given set containing $GF(3)$ was considered first by Whittle [71, 72]. Specifically he showed that if $\mathcal{F}$ is a set of fields containing $GF(3)$ and $\mathcal{M}$ is a class of matroids that are representable over all fields in $\mathcal{F}$, then $\mathcal{M}$ is the class of matroids representable over $GF(3)$ and $GF(q)$ for some $q \in \{2, 3, 4, 5, 7, 8\}$. Furthermore he categorized the matroids that are representable over a set of fields containing $GF(3)$ into four basic classes, regular, near-regular, dyadic and $\sqrt[6]{1}$-matroids. Apart from regular matroids, algebraic characterizations have also been given for the last three classes of matroids [73].

Let $\mathcal{Q}(\alpha)$ be the field obtained by extending the rationals by the transcendental α. A matrix over $\mathcal{Q}(\alpha)$ is *near-regular* if all of its non-zero subdeterminants are in $\{\pm(\alpha^i - 1)\alpha^j : i, j \in \mathbb{Z}\}$. A matroid is *near-regular* if it can be represented by a near-regular matrix. In [71] Whittle proved an algebraic characterization for near-regular matroids. Specifically he showed that a matroid M is ternary and representable over both the rationals and $GF(4)$ if and only if M is ternary and representable over both $GF(4)$ and $GF(5)$. Equivalently M is representable over all fields except possibly $GF(2)$ if and only if M is near-regular. The class of near-regular matroids contains the class of regular matroids and is the intersection of $\sqrt[6]{1}$ and dyadic matroids (Figure 3.4).

A matrix over $\mathbb{Q}$ is *dyadic* if all of its non-zero subdeterminants are in $\{0, \pm 2^i : i \in \mathbb{Z}\}$. A matroid is *dyadic* if it can be represented by a dyadic matrix. An algebraic characterization for dyadic matroids very similar to that of near regular matroids was presented by Whittney in [72]. More precisely, a matroid M is dyadic if and only if M is representable over both $GF(3)$ and $\mathbb{Q}$ if and only if M is representable over $GF(3)$ and $GF(5)$.

A $\sqrt[6]{1}$-*matrix* is a matrix over the complex numbers such that all of its subdeterminants are complex sixth roots of unity. A $\sqrt[6]{1}$-*matroid* is a matroid that can be represented over the complex numbers by a $\sqrt[6]{1}$-matrix. The class of $\sqrt[6]{1}$-matroids is precisely the class of matroids representable over $GF(3)$ and $GF(4)$.

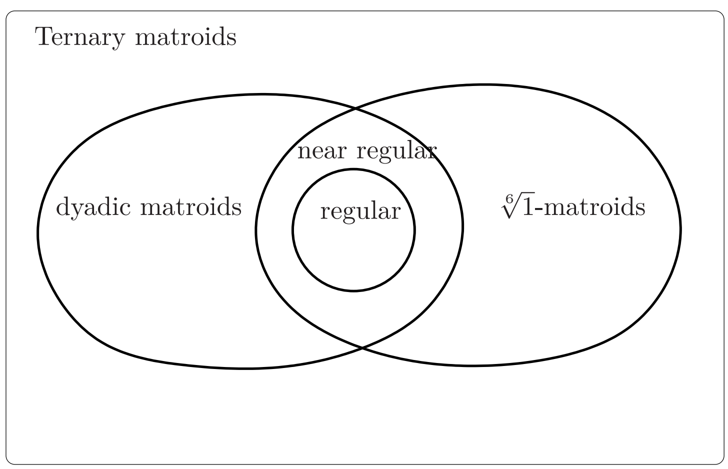


Figure 3.2: The relationships between near regular, $\sqrt[6]{1}$ and dyadic matroids.

Attempts to extend the problem of characterizing a class of matroids that are representable over all fields in a given set of fields containing $GF(4)$ showed that there are an infinite number of classes that are the class of matroids representable over some set of fields containing $GF(4)$. Regular, near-regular, $\sqrt[6]{1}$ and dyadic are all classes of matroids that are obtained by taking representations whose subdeterminants belong to some subgroup of the multiplicative group of a field. Based on this observation, Semple and Whittle developed a theory of matroid representation over algebraic structures called *partial fields* [48]. Vertigan worked on this theory and obtained many interesting results including the following. A matrix over $\mathbb{R}$ is *golden mean* if all of its non-zero subdeterminants are in $\{\pm r^i(1-r)^j : i,j \in \mathbb{Z}\}$, where r denote a real root of the polynomial x^2-x-1 and the other root is $1-r$. A matroid is *golden mean* if it can be represented over the reals by a golden mean matrix. The following characterization for golden mean matroids was provided by Vertigan, but its proof was published by Pendavingh and Van Zwam [43]. A matroid is golden mean if and only if it is representable over both $GF(4)$ and $GF(5)$. Pendavingh and Van Zwam extended the theory of matroids representable over partial fields further giving new results along with new proofs of Whittle's results. In particular they showed that when a matroid is representable over a partial field then it is representable over some field [43, 42].

Excluded minor characterizations over sets of fields

Another question that arises is whether we can derive excluded minor characterizations for classes of representable matroids over all fields in a given set of fields. This open problem has led to a conjecture that generalizes Rota's conjecture as follows. If $\mathcal{F}$ is a set of fields, at least one of which is finite, then the class of matroids representable over all fields in $\mathcal{F}$ has a finite set of excluded minors.

In a series of papers [59, 60, 61, 63] Tutte provided the following excluded minor characterization for the class of regular matroids.

Theorem 10. *A matroid is regular if and only if it has no minor isomorphic to* $U_{2,4}$ F_7, F_7^*.

For the classes of near regular, $\sqrt[6]{1}$ and dyadic matroids there are not yet excluded minor characterizations. For the class of near regular matroids, Geelen announced that, by adapting the techniques of the proof of the excluded-minor characterization of $GF(4)$-representable matroids, one could determine the complete list of excluded minors. However, the proof for the excluded-minor characterization of near regular matroids was published by Hall, Mayhew, and Van Zwam in 2009.

Theorem 11. *A matroid is near-regular if and only if it has no minor isomorphic to any of* $U_{2,5}, U_{3,5}, F_7, F_7^*, F_7^-, (F_7^-)^*, AG(2,3)\backslash e, (AG(2,3)\backslash e)^*, \Delta_3(AG(2,3)\backslash e)$, *or* P_8.

As regards the class of $\sqrt[6]{1}$-matroids, its excluded minors will follow from those for ternary and quaternary matroids, since they are representable over both $GF(3)$ and $GF(4)$ fields [18]. Similarly, since dyadic matroids are representable over $GF(3)$ and $GF(5)$, the excluded minors for dyadic matroids would follow from those of ternary and $GF(5)$-representable matroids. In [35], Oxley conjectures the list of excluded minors for dyadic matroids, however a proof of this has not been published yet.

Decomposition theorems

A very deep and important result in matroid theory is Seymour's decomposition theorem for regular matroids [50]. According to the latter theorem, each regular matroid can be obtained from graphic matroids, cographic matroids and copies of R_{10} by taking 1-,2- and 3-sums. Seymour's regular matroid decomposition theorem not only provided a polynomial time algorithm for recognizing regular matroids, but it also led to a polynomial time algorithm for determining whether a real matrix is totally unimodular. The matrix A_{10}, which is given below, is a representation matrix for matroid R_{10} over $GF(3)$.

$$A_{10} = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

It is of desire to obtain similar decomposition theorems for other classes of matroids that are representable over some finite field or even for classes of matroids that are representable over all fields in a given set of fields. These theorems would lead to algorithms that determine for each matroid M in the class, given by an independence oracle, in polynomial time in $|E(M)|$ whether M belongs to the class. The next result, whose proof is unpublished, has eliminated the hopes for many classes of matroids. Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be finite fields. If either both $\mathbb{F}_1$ and $\mathbb{F}_2$ are not prime, or both $\mathbb{F}_1$ and $\mathbb{F}_2$ are prime with at least five elements, then there is no polynomial algorithm for recognizing the class of matroids representable over both $\mathbb{F}_1$ and $\mathbb{F}_2$. However, there are still hopes for classes of matroids that are not included. In particular, it is conjectured that the classes of near regular, $\sqrt[6]{1}$ and dyadic matroids are polynomially recognizable [73]. The first step towards their recognition is to determine the classes of matroids that play a role analogous to that played by graphic and cographic matroids in Seymour's regular decomposition theorem.

3.4 Graphic matroids

In this section, we present well-known results and open problems concerning the fundamental classes of matroids that derive from graphs, signed graphs and biased graphs.

3.4.1 Matroids arise from graphs

One of the two fundamental classes of matroids that appear in [70] arises from graphs as follows.

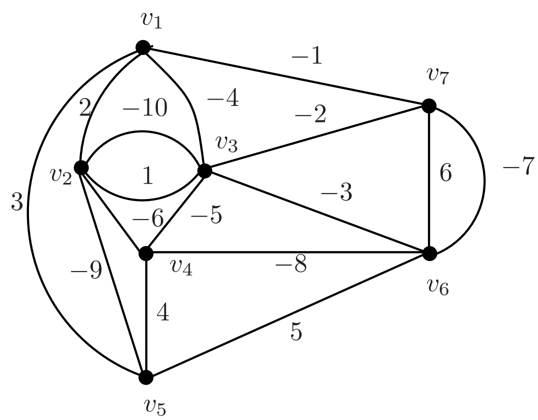


Figure 3.3: A graph

Theorem 12. *Let $E(G)$ be the set of edges of a graph G and $\mathcal{C} \subseteq 2^{E(\Sigma)}$ is the collection of edge sets of cycles of G, then $M(G) = (E(G), \mathcal{C})$ is a matroid on $E(G)$ with circuit family $\mathcal{C}$ called the cycle matroid of G.*

A matroid that is isomorphic to the cycle matroid of a graph is called *graphic*. For a graph G we denote the dual of the cycle matroid of G by $M^*(G)$. This matroid is called the *bond matroid of G* or the *cocycle matroid of G*. A matroid which is isomorphic to the cocycle matroid of a graph is called *cographic*. Several operations and results for graphs were translated to corresponding operations and results for graphic matroids. Menger's Theorem for graphs was generalized to matroids by Tutte' s Linking Theorem. Tutte's Wheel Theorem for graphs was proved for matroids also by Tutte establishing that wheels and whirls are the only 3-connected matroids for which any one element deletion or contraction is not 3-connected. Moreover, Tutte decomposed a 2-connected graph into 3-connected graphs, cycles and cocycles in his book 'Connectivity in graphs' [64]. This decomposition result was proved for matroids by Cunningham and Edmonds establishing the existence of a unique 2-sum decomposition of a connected matroid into 3-connected matroids, circuits and cocircuits [9].

3.4.2 Representation matrices for graphic matroids

Given a graph G the incidence matrix A_G is a representation matrix of the associated cycle matroid over $GF(2)$. Moreover, if we assume that $[I|D]$ is a standard representation matrix of $M(G)$, that is obtained from A, then the columns of the compact representation matrix D are characteristic vectors of cycles in G. As a result, there is computational method for the conctruction of $D = (d_{i,j}) \in GF(2)$ as follows. Given a basis $B = \{e_1, e_2, \ldots, e_r\}$ of $M(G)$ and $e_j \in E(M(G)) - B$ the entry $d_{i,j} = 1$ if $e_i \in C(B, e_j)$ $(i \in \{1, \ldots, r\})$ and 0 otherwise.

Example 3.4.1. *The graph G in Figure 3.3 has cycle matroid $M(G)$ with ground set $E(M(G)) = \{1,2,3,4,5,6,-1,-2,-3,-4,-5,-6,-7,-8,-9,-10\}$ and circuit family $\mathscr{C}(M(G))$ the set of edge sets of G. The incidence matrix of G is given below.*

$$A_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{array} \\ \left[\begin{array}{cccccccccccccccc} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Applying elementary row operations in $GF(2)$ and column interchanges to A_G we obtain the following standard representation matrix $[I|D]$ of $M(G)$. The set $B = \{1,2,3,4,5,6\} \in \mathscr{B}(M(G))$.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{array} \\ \left[\begin{array}{cccccc|cccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \end{array}$$

The matrix D is a compact representation matrix of $M(G) = M[A_G]$ over GF(2) field.

$$D = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Given a connected directed graph $\overrightarrow{G}$, applying elementary row operations and column interchanges to the incidence matrix of $\overrightarrow{G}$ we obtain a compact representation matrix of $M(G)$ over $GF(3)$. This representation matrix over $\mathbb{R}$ is the network matrix N associated with $\overrightarrow{G}$. A computational method can be applied in order to obtain the entries of $N = (n_{i,j}) \in GF(3)$ as follows. Let T be a tree of $\overrightarrow{G}$ and let $C = C(T, e_j)$ be the fundamental cycle of e_j with respect to T. If C is a loop e_j, then the corresponding column of N is zero. Now assume that C is not a loop and let the edges of $C - \{e_j\}$, in cyclic order, be $e_1, e_2, \ldots, e_n$. Now cyclically traverse C beginning with e_j and for each $e_i, i \in \{1, \ldots, n\}$ let $n_{i,j}$ be 1 if the direction of traversal agrees with the direction of e_j or -1 otherwise.

Example 3.4.2. *Consider the directed graph $\overrightarrow{G}$ in Figure 2.2 and the cycle matroid $M(G)$ with ground set $E(M(G)) = \{1, 2, 3, 4, 5, 6, -1, -2, -3, -4, -5, -6, -7, -8, -9, -10\}$ and circuit family $\mathscr{C}(M(G))$ the set of edge sets of G. The incidence matrix $A_{\overrightarrow{G}}$ of $\overrightarrow{G}$ is given below.*

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Applying elementary row operations in $GF(3)$ and column interchanges to $A_{\overrightarrow{G}}$ we obtain the following standard representation matrix $[I|N]$ of $M(G)$.

$$\begin{array}{cc} & \begin{array}{cccccc|cccccccccc} 1&2&3&4&5&6&-1&-2&-3&-4&-5&-6&-7&-8&-9&-10 \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} & \left[\begin{array}{cccccc|cccccccccc} 1&0&0&0&0&0&0&-1&-1&-1&-1&0&0&0&0&-1\\ 0&1&0&0&0&0&0&-1&-1&-1&-1&-1&0&0&1&0\\ 0&0&1&0&0&0&1&1&1&0&1&1&0&0&-1&0\\ 0&0&0&1&0&0&0&0&0&0&-1&-1&0&1&0&0\\ 0&0&0&0&1&0&-1&-1&-1&0&0&0&0&-1&0&0\\ 0&0&0&0&0&1&-1&-1&0&0&0&0&-1&0&0&0 \end{array}\right] \end{array}$$

The matrix N is the network matrix associated with $\overrightarrow{G}$ and a compact representation matrix of $M(G)$ over GF(3) field.

$$N = \begin{array}{cc} & \begin{array}{cccccccccc} -1&-2&-3&-4&-5&-6&-7&-8&-9&-10 \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array} & \left[\begin{array}{cccccccccc} 0&-1&-1&-1&-1&0&0&0&0&-1\\ 0&-1&-1&-1&-1&-1&0&0&1&0\\ 1&1&1&0&1&1&0&0&-1&0\\ 0&0&0&0&-1&-1&0&1&0&0\\ -1&-1&-1&0&0&0&0&-1&0&0\\ -1&-1&0&0&0&0&-1&0&0&0 \end{array}\right] \end{array}$$

As regards connectivity for graphic matroids, the concept of n-connectivity for matroids was introduced by Tutte in order to generalize the corresponding concept for graphs and to incorporate duality into the theory [65]. The importance of Tutte's definition for n-connectivity was highlighted by the fact that the n-connectivity of a graph and its cycle matroid coincide. Whitney proved that cycle matroids of 3-connected graphs are uniquely representable and gave necessary and sufficient conditions for two graphs to have isomorphic cycle matroids in his 2-Isomorphism Theorem [68]. Tutte proved that graphic matroids are representable over every field while Whitney showed that the intersection of the class of cographic matroids with the class of graphic matroids is the class of matroids that arise from planar graphs [69].

3.4.3 Characterizations and decomposition theorems

The most important results concerning the class of graphic matroids are excluded minor characterizations and decomposition theorems that lead to polynomial time recognition algorithms. Excluded minor characterizations were given by Tutte [61, 63] and Bixby [3].

Theorem 13 (Tutte [61])**.** *A matroid is graphic if and only if it has no minor isomorphic to $U_{2,4}$ F_7, F_7^*, $M^*(K_{3,3})$ and $M^*(K_5)$.*

Theorem 14 (Tutte [63])**.** *A binary matroid is graphic if and only if it has no minor isomorphic to* F_7, F_7^*, $M^*(K_{3,3})$ *and* $M^*(K_5)$.

Theorem 15 (Tutte [61])**.** *A regular matroid is graphic if and only if it has no minor isomorphic to* $M^*(K_{3,3})$ *and* $M^*(K_5)$.

A structural characterization for graphic matroids was given by Seymour in [51]. Seymour's characterization was based on the fact that when the cycle matroid $M(G)$ of a graph G is different from a matroid M on $E(G)$ with the same rank, then there is a vertex of G whose star is not a cocircuit of M. Characterizations which were based on properties of cocircuits were given by Fournier [15], Mighton [34], Tutte [62] and others. In 1974, Fournier characterized the class of graphic matroids by proving that a matroid is graphic if and only if for any three cociruits with a nonempty intersection there is one that separates the other two [15]. In 2008, Mighton gave two necessary and sufficient conditions for a binary matroid to be graphic, where the one condition was a reduction of Fournier's condition to fundamental cocircuits with respect to a basis and the other was the bridge-separability property of each of the fundamental cocircuits [34]. Mighton's characterization in contrary with Fournier's was a polynomial time algorithm for testing graphicness. In 2011, Geelen and Gerards provided another characterization for the class of graphic matroids based on the existence of a solution of a linear system when a set of fundamental cocircuits with respect to a basis of a binary matroid correspond to that of a graphic matroid [17]. In 2010 Wagner presented a simpler proof of Mighton's characterization for graphic matroids using Tutte's decomposition theorem for graphic matroids [67]. Furthermore a characterization in terms of circuits was given by Wagner in [66].

The first decomposition theorem characterizing the class of graphic matroids was proved by Tutte and was based on the deletion of a cocircuit (Theorem 51). More precisely, given a cocircuit Y of a binary matroid M, M is graphic if and only if Y is bridge-separable and for each B bridge of Y in M the matroid $M.(B \cup Y)$ is graphic. The decomposition theory that he developed for graphic matroids led to the first algorithm for determining whether a binary matroid is graphic [62]. Furthermore he pioneered the recursive approach of testing a matroid for a property by testing some well defined minors.

All the above characterizations, apart from Fournier's, imply algorithms for testing graphicness. Polynomial time algorithms for recognizing graphic matroids were also presented by Cunningham in [8] and Bixby and Cunningham in [5]. Specifically Bixby and and Cunningham gave a linear time algorithm using Tutte's decomposition theory for graphic matroids. A few years later, two almost linear time algorithms were published by Bixby and Wagner in [6] and by Fujishige in

[16].

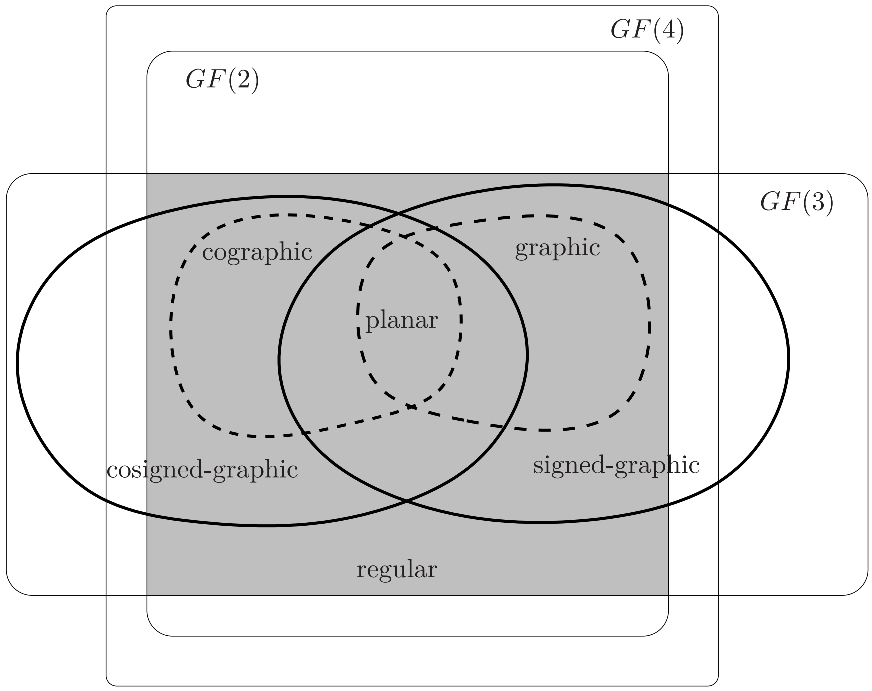


Figure 3.4: The relationships between classes of matroids.

3.5 Signed-graphic and bias matroids

It is conjectured that bias matroids are bulding blocks for the class of near-regular matroids in a similar way that graphic matroids are for regular matroids [73]. Pagano has essentially characterized the class of bias matroids that are near-regular in [37] and it seems clear that there is a polynomial algorithm for recognizing this class. The fundamental classes of signed-graphic and bias matroids are defined in the next subsections.

3.5.1 Matroids arise from signed graphs

Signed graphs, that is graphs whose edges have been attributed a sign, were defined by Harary in [27] in 1954. Three decades later, Zaslavsky studied signed graphs and their properties in his article 'Signed graphs'. Moreover, in the same article he introduced signed-graphic matroids that is a class of matroids that arises from signed graphs.

Theorem 16. *Let $E(\Sigma)$ be the set of edges of a signed graph Σ and $\mathscr{C}$ be the family of minimal edge sets inducing a subgraph that is either:*

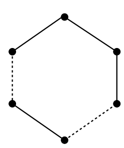
(a) a positive cycle, or

(b) two negative cycles which have exactly one common vertex, or

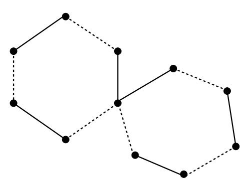
(c) two vertex-disjoint negative cycles connected by a path that has no common vertex with the cycles apart from its endvertices.

Then $M(\Sigma) = (E(\Sigma), \mathscr{C})$ is a matroid on $E(\Sigma)$ with circuit family $\mathscr{C}$ and it is called the signed-graphic *matroid of Σ.*

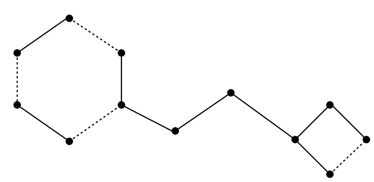
The connected subgraphs of Σ which are of the types $(a) - (c)$ in the above definition are called *circuits of* Σ. Moreover, the subgraphs of Σ which are described in cases (b) and (c) are called *type I* and *type II handcuff* respectively. In figures of signed graphs, dashed lines are used to depict negative edges while solid lines are used to depict positive edges. The three types of circuits of a signed-graphic matroid $M(\Sigma)$ are depicted in Figure 3.5.



(a) Positive cycle



(b) Tight handcuff



(c) Loose handcuff

Figure 3.5: Circuits in a signed graph Σ.

Given a signed graph Σ and a subset A of $E(M(\Sigma))$, it follows that A corresponds to a subset of edges in Σ. Frequently, we shall refer to the induced subgraph of the edgeset of A in Σ, instead of the edgeset of A.

Zaslavsky defined many operations on signed graphs such as switching, deletion and contraction. The operations which are performed at a signed graph leaving the matroid unchanged were presented in [55, 74, 76]. These operations are mentioned in the next proposition which appears in [40].

Proposition 11. *Let Σ be a signed graph. If Σ':*

(i) is obtained from Σ by replacing any number of negative loops by half-edges and vice versa, or

(ii) *is obtained from* Σ *by switchings, or*

(iii) *is the twisted graph of* Σ *about* (u, v) *with* Σ_1, Σ_2 *the twisting parts of* Σ*, where* Σ_1 *(or* Σ_2*) is balanced or all of its negative cycles contain* u *and* v*,*

then $M(\Sigma) = M(\Sigma')$*.*

Necessary conditions under which a signed-graphic matroid is graphic are established in the next proposition. These results can be found in [55, 74, 76].

Proposition 12. *Let* Σ *be a signed graph. If* Σ

(i) *is balanced, or*

(ii) *has no negative cycles other than joints, or*

(iii) *has a balancing vertex,*

then $M(\Sigma)$ *is graphic.*

The class of signed-graphic matroids is closed under the operations of contraction and deletion as it is stated in the following theorem.

Theorem 17. *If* Σ *is a signed graph and* $S \subseteq E(\Sigma)$*, then* $M(\Sigma \backslash S) = M(\Sigma) \backslash S$ *and* $M(\Sigma / S) = M(\Sigma)/S$*.*

3.5.2 Representation matrices for signed-graphic matroids

Zaslavsky proved that signed-graphic matroids are representable over any field of characteristic not equal to 2 [74]. Therefore what was left was to determine when a signed-graphic matroid is representable over fields of characteristic two. The representability of signed-graphic matroids over $GF(2)$ and $GF(4)$ fields was studied by Gerard [25], Pagano [37] and Slilaty and Qin [55].

Representation matrices for signed-graphic matroids were presented by Papalamprou and Pitsoulis in [38, 39, 44]. In the following, we present two theorems which appear in [39, 44].

Theorem 18 (Papalamprou and Pitsoulis [39])**.** *Given a signed graph* Σ

(i) A_Σ *is a representation of* $M(\Sigma)$ *in* $GF(3)$*, and*

(ii) $A_{\overrightarrow{\Sigma}}$ *is a representation of* $M(\Sigma)$ *in* $\mathbb{R}$*, where* $\overrightarrow{\Sigma}$ *is a bidirected graph obtained by an arbitrary orientation of* Σ*.*

A matrix is called *integral* if its entries are integers.

Theorem 19 (Papalamprou and Pitsoulis [39]). *Let B be an integral binet matrix and $M(\Sigma)$ be the signed-graphic matroid represented by B over $\mathbb{R}$. The matrix $B' = B \bmod 3$ is a representation matrix of $M(\Sigma)$ over $GF(3)$.*

In the following example we describe how, given a signed graph, we derive a compact representation matrix for the associated signed-graphic matroid.

Example 3.5.1. *Consider the incidence matrix A_Σ of the signed graph in figure 2.3 which is given below*

$$
\begin{array}{c}
\\
v_1\\ v_2\\ v_3\\ v_4\\ v_5\\ v_6\\ v_7
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc}
\scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 5 & \scriptstyle 6 & \scriptstyle 7 & \scriptstyle -1 & \scriptstyle -2 & \scriptstyle -3 & \scriptstyle -4 & \scriptstyle -5 & \scriptstyle -6 & \scriptstyle -7 & \scriptstyle -8 & \scriptstyle -9
\end{array}\\
\begin{bmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1\\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0\\
0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0
\end{bmatrix}
\end{array}
$$

Applying elementary row operations over $\mathbb{R}$ and column interchanges to A_Σ we obtain the following standard representation matrix $[I|D]$ of $M(\Sigma)$ over $\mathbb{R}$.

$$
\begin{array}{c}
\\
1\\ 2\\ 3\\ 4\\ 5\\ 6\\ 7
\end{array}
\begin{array}{c}
\begin{array}{ccccccc|ccccccccc}
\scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 5 & \scriptstyle 6 & \scriptstyle 7 & \scriptstyle -1 & \scriptstyle -2 & \scriptstyle -3 & \scriptstyle -4 & \scriptstyle -5 & \scriptstyle -6 & \scriptstyle -7 & \scriptstyle -8 & \scriptstyle -9
\end{array}\\
\left[\begin{array}{ccccccc|ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & -1 & -1 & 1 & -1 & -1 & -2 & 0 & 1\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 2 & 0 & -1\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
\end{array}
$$

The matrix D is a compact representation matrix of $M(\Sigma)$ over $\mathbb{R}$. Moreover, the matrix $D' = D \bmod 3$ with entries in $\{0, 1, -1\}$ is a representation matrix for $M(\Sigma)$ over $GF(3)$.

$$D' = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \end{array} \\ \left[\begin{array}{ccccccccc} 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \end{array}$$

By Theorem 18, the incidence matrix of the bidirected graph which is obtained by an orientation of the signed graph Σ represents $M(\Sigma)$ over $\mathbb{R}$. In the following example, we describe how, given a bidirected graph, we derive a compact representation matrix for the associated signed-graphic matroid.

Example 3.5.2. *Consider an orientation of the signed graph Σ in Figure 2.3. The bidirected graph which is obtained is depicted in Figure 2.5 and its incidence matrix $A_{\overrightarrow{\Sigma}}$ is given below.*

$$A_{\overrightarrow{\Sigma}} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \end{array} \\ \left[\begin{array}{cccccccccccccccc} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] \end{array}$$

Applying elementary row operations over $\mathbb{R}$ and column interchanges to $A_{\overrightarrow{\Sigma}}$ we obtain the following standard representation matrix $[I|B]$ of $M(\Sigma)$ over $\mathbb{R}$.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc|ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \end{array} \\ \left[\begin{array}{ccccccc|ccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & -1 & -1 & 1 & -1 & -1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \end{array}$$

The matrix B is the binet matrix associated with $\overrightarrow{\Sigma}$ and a compact representation matrix of $M(\Sigma)$ over $\mathbb{R}$. Furthermore, the matrix $B' = B$ mod 3 with entries in $\{0, 1, -1\}$ is a representation matrix for $M(\Sigma)$ over $GF(3)$.

$$B' = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

A combinatorial algorithm to compute the entries of a binet matrix associated with a bidirected graph in the same spirit as the one described for network matrices in Example 3.4.2 was published independently by Appa and Kotnyek [1] and Zaslavsky [77].

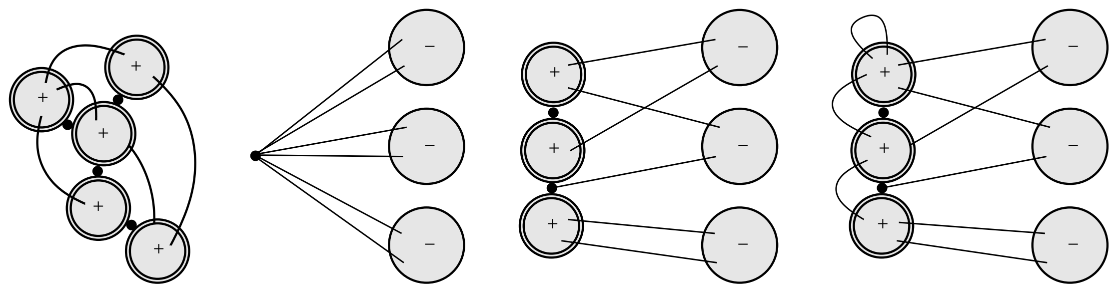
3.5.3 From signed-graphic matroids to signed graphs

The elements of a cocircuit of a signed-graphic matroid were characterized graphically by Zaslavsky as follows.

Theorem 20 (Zaslavsky [74]). *Given a signed graph Σ and its corresponding matroid $M(\Sigma)$, $Y \subseteq E(\Sigma)$ is a cocircuit of $M(\Sigma)$ if and only if Y is a minimal set of edges whose deletion increases the number of balanced components of Σ.*

A cocircuit Y of a matroid M is called *graphic* if $M \backslash Y$ is graphic, otherwise it is called *non-graphic*. The set of edges of a signed graph that corresponds to a cocircuit of the associate signed-graphic matroid is called *bond*. It is important to note that the deletion of a bond Y from a connected signed graph Σ results in a signed graph $\Sigma \backslash Y$ with exactly one balanced component due to the minimality of Y. Hence $\Sigma \backslash Y$ is either connected and balanced or consists of one balanced component and one or more unbalanced components. Bonds can be classified into four types according to the signed graph obtained upon their deletion. Let us assume first that Σ is an unbalanced signed graph. If $\Sigma \backslash Y$ is a connected and balanced signed graph, then Y is called a *balancing bond*. Otherwise $\Sigma \backslash Y$ consists of one balanced component and one or more unbalanced components. If there are edges of Y with both endvertices (or the unique endvertex if they are joints) at the balanced component of $\Sigma \backslash Y$ then we say that Y is a *double bond*. On the other hand, if every edge of Y has one endvertex at the balanced component of

$\Sigma\backslash Y$ and one at an unbalanced component, then Y is called an *unbalancing bond*. In case the balanced component is empty of edges then we say that Y is a *star bond*. Let us assume now that Σ is a balanced signed graph, then $\Sigma\backslash Y$ consists of two balanced components and Y is called unbalancing bond. The types of bonds which were defined above are depicted in Figure 3.6, where a single line is used to represent connected underlying subgraphs while a double line 2-connected underlying subgraphs. The signs indicate whether the subgraphs are balanced (+) or not (−).



(a) balancing bond (b) star bond (c) unbalancing bond (d) double bond

Figure 3.6: Bonds in signed graphs

Let Y be a double bond of a connected signed graph Σ, then the edges of Y can be partitioned into two parts, the *unbalancing* and the *balancing* part. The unbalancing part contains the edges of Y which have exactly one end-vertex in the balanced component of $\Sigma\backslash Y$, while the balancing part contains the edges that have both their end-vertices or their unique end-vertex (if they are joints) in the balanced component. A further classification of bonds is based on whether the matroid $M(\Sigma)\backslash Y$ is connected or not. If $M(\Sigma)\backslash Y$ has more than one connected components then Y is called *separating bond* of Σ, otherwise we say that Y is a *non-separating bond*.

In the following theorem, the elementary separators of a signed-graphic matroid are characterized with respect to the edge set of the corresponding signed graph.

Theorem 21 (Zaslavsky [76]). *Let Σ be a connected signed graph. The elementary separators of $M(\Sigma)$ are the edge sets of each outer block and the core, except that when the core is a B-necklace each block in the B-necklace is also an elementary separator.*

If M is signed-graphic and Σ is a signed graph such that $M = M(\Sigma)$ then a *separator* of $\Sigma\backslash Y$ is a bridge of Y in M. Let us suppose that B is a separator of $\Sigma\backslash Y$ and Σ_i is the connected component of $\Sigma\backslash Y$ such that $B \subseteq \Sigma_i$, then we denote by $C(B, v)$, where $v \in V(B)$, the connected component of $\Sigma_i\backslash B$ having v as a vertex. Furthermore, we denote by $Y(B, v)$ the set of all elements in Y with either one end-vertex or both end-vertices at $C(B, v)$. A vertex v of a separator B of the signed graph $\Sigma\backslash Y$ such that $Y(B, v) \neq \emptyset$ is called a *vertex of attachment* of B. For a separator B and a vertex $v \in V(\Sigma[B])$, if Σ_1 is the component of $\Sigma\backslash Y$ such that $\Sigma[B] \subseteq \Sigma_1$, we define $F(B, v) = \Sigma_1\backslash C(B, v)$. Let us assume now that Σ is a balanced signed graph and Y is a bond of Σ then the separators of $\Sigma\backslash Y$ are depicted in Figure 3.7.

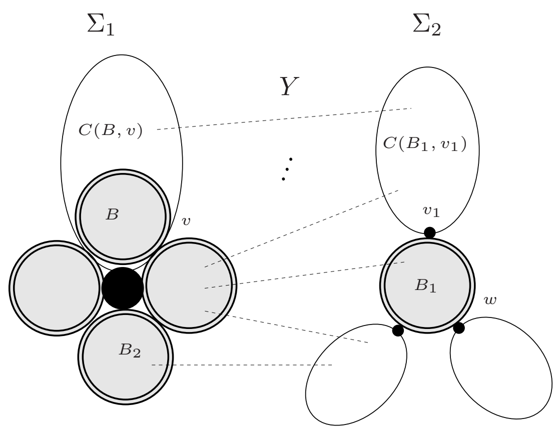


Figure 3.7: The separators of $\Sigma\backslash Y$

k-biconnectivity of a signed graph is related to k-connectivity of the associated signed-graphic matroid as follows.

Proposition 13 (Slilaty and Qin [56]). *Let Σ be a connected and unbalanced signed graph.*

(i) If (A, B) is a k-biseparation of Σ, then (A, B) is a k-separation of $M(\Sigma)$.

(ii) If (A, B) is an exact k-separation of $M(\Sigma)$ with connected parts, then (A, B) is a k-biseparation of Σ.

3.5.4 Matroids arise from biased graphs

Biased graphs and gain graphs were defined by Zaslavsky in his article 'Biased graphs I', where he studied also their structural properties [75]. The bias matroid or frame matroid $F(\Omega)$ of a biased graph Ω was defined by Zaslavsky as well in [76]. Moreover, he provided cryptomorphic definitions and proved that the bias matroids of biased graphs are dyadic.

Definition 3.5.1. *Let $E(\Omega)$ be the set of edges of a biased graph Ω and $\mathscr{C}$ be the family of minimal edge sets inducing a subgraph that is either:*

(i) a balanced cycle, or

(ii) two unbalanced cycles which have exactly one common vertex, or

(iii) two vertex-disjoint unbalanced cycles connected by a path that has no common vertex with the cycles apart from its endvertices, or

(iii) a theta graph with all cycles unbalanced

Then $F(\Omega) = (E(\Omega), \mathscr{C})$ is a matroid on $E(\Omega)$ with circuit family $\mathscr{C}$ and it is called the bias *or* frame *matroid of Ω.*

The biased graph which represents uniquely a signed-graphic matroid induces a signed graph as shown in the next proposition.

Proposition 14. *Let the biased graph $\Omega = (G, \Gamma)$ represent uniquely the bias matroid $F(\Omega)$ up to isomorphism. If there is a signed graph $\Sigma = (G, \sigma)$ such that $F(\Omega) \cong M(\Sigma)$ then Σ is a unique representation of $F(\Omega)$ up to isomorphism.*

Proof. Assume on the contrary that there is a signed graph $\Sigma' = (G_{\Sigma'}, \sigma')$ such that $\Sigma' \ncong \Sigma$ which represents $F(\Omega)$. By assumption the signed graph Σ' represents $M(\Sigma)$ therefore $M(\Sigma) \cong M(\Sigma')$. From $\Sigma' = (G_{\Sigma'}, \sigma')$ we construct $\Omega' = (G_{\Sigma'}, \Gamma')$ where Γ' is the set of positive cycles of Σ' and therefore $M(\Sigma') \cong F(\Omega')$. Since $\Sigma \ncong \Sigma'$ we have that $\Omega \ncong \Omega'$. However, Ω' represents $F(\Omega)$ as $F(\Omega') \cong M(\Sigma')$ $\cong M(\Sigma) \cong F(\Omega)$, which is a contradiction. $\square$

The structure of a biased graph that represents uniquely a bias matroid is described in Theorem 2 of [53]. The latter theorem is stated for signed-graphic matroids as follows.

Proposition 15. *Let Σ be a 3-connected signed graph without balanced loops, loose edges and isolated vertices. If Σ contains three vertex-disjoint unbalanced cycles at most one of which is a loop, then Σ is a unique representation for $M(\Sigma)$.*

Proof. Let $\Sigma = (G, \sigma)$ be a signed graph that satisfies the hypothesis of the lemma. We construct the bias graph $\Omega = (G, \Gamma)$ from Σ such that Γ is the set of positive cycles of Σ. Then $F(\Omega) = M(\Sigma)$ and by Proposition 14, Σ is a unique representation of $M(\Sigma)$. $\square$

3.5.5 Structural results and decomposition theorems

The class of signed-graphic matroids contains the class of graphic matroids and the class of even-cycle matroids of graphs, while it forms a subclass of bias matroids. Thereby many results concerning the class of graphic matroids were generalised to results for the classes of signed-graphic and bias matroids. The structure of biased graphs whose bias matroids have a unique graphical representation was determined by Slilaty [53] generalising Whitney's famous result for graphs. Moreover, Whitney's Theorem which states that planarity is the necessary and sufficient condition for a connected cographic matroid to be graphic was generalized also by Slilaty in [52]. More precisely, he showed that projective-planarity is the necessary and sufficient condition for a connected cographic matroid to be signed-graphic. As a result, the cographic matroids of the 29 vertically 2-connected graphs $G_1, \ldots, G_{29}$, which are excluded minors for projective-planar graphs, were proved to be among the excluded minors for the class of signed-graphic matroids.

In 2009 Qin, Slilaty and Zhou provided the complete list of regular excluded minors for signed-graphic matroids and an excluded minor characterization for the class of regular signed-graphic matroids [45].

Signed-graphic matroids and bias matroids were studied in terms of structural properties of the signed graphs representing them by Pagano in his dissertation [37]. Specifically, he characterized signed graphs whose matroids are binary as follows..

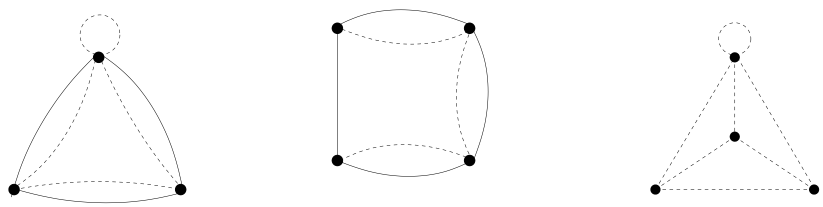
Theorem 22 (Pagano [37])**.** *Let Σ be a connected signed graph and Σ' be the signed graph obtained from Σ upon the contraction of any balanced blocks. Then $M(\Sigma)$ is binary if and only if Σ' has no two vertex-disjoint negative cycles.*

The proof of the above theorem was based on Tutte's excluded minor characterization of binary matroids. Furthermore Pagano proved an excluded minor characterization of signed graphs whose matroids are quaternary by showing that the matroids of the signed graphs $\pm C_3^{(1)}$, $\pm C_4 \backslash e$ and $-K_4^{(1)}$ in Figure 3.8 are excluded minors of the class of quaternary signed-graphic matroids.

Signed graphs whose bias matroids are representable over $GF(2)$ and $GF(4)$ fields were decomposed by Slilaty and Qin [55] combining Pagano's and Gerard's results. The following theorem presents the k-sum decomposition of the class of binary signed-graphic matroids.

Theorem 23 (Pagano [37], Slilaty, Qin [55])**.** *If Σ is connected and $M(\Sigma)$ is binary, then either*

(i) Σ is balanced,



(a) $\pm C_3^{(1)}$ (b) $\pm C_4\backslash e$ (c) $-K_4^{(1)}$

Figure 3.8: Excluded minors of quaternary signed-graphic matroids

(ii) Σ *is joint unbalanced,*

(iii) Σ *has a balancing vertex,*

(iv) Σ *is tangled, or*

(v) $\Sigma = \Sigma_1 \oplus_k \Sigma_2$ *for* $k \in \{1, 2\}$ *where each* $M(\Sigma_i)$ *is binary.*

Also, if Σ *is a connected signed graph that satisfies one of (i)-(iv), then* $M(\Sigma)$ *is binary.*

Given a signed graph Σ the signed graph which is obtained after removing any joints is denoted by $\Sigma\backslash J_\Sigma$. The following theorem describes the k-sum decomposition of quaternary signed-graphic matroids. The signed graph T_6, which is mentioned in Theorem 24, is depicted in Figure 7.1.

Theorem 24 (Pagano [37], Slilaty and Qin [55])**.** *If* Σ *is connected and* $M(\Sigma)$ *is quaternary, then either*

(i) $M(\Sigma)$ *is binary,*

(ii) $\Sigma\backslash J_\Sigma$ *has a balancing vertex,*

(iii) $\Sigma\backslash J_\Sigma$ *is cylindrical,*

(iv) $\Sigma\backslash J_\Sigma \cong T_6$*, or*

(v) $\Sigma\backslash J_\Sigma = \Sigma_1 \oplus_k \Sigma_2$ *for* $k \in \{1, 2, 3\}$ *where each* $M(\Sigma_i)$ *is quaternary.*

Also, if Σ *is a connected signed graph that satisfies one of (i)-(iv), then* $M(\Sigma)$ *is quaternary.*

Theorems 23 and 24 provide also the signed graphs that represent binary and quaternary signed-graphic matroids respectively. The following result which is essential for the decomposition of quaternary signed-graphic matroids states that the class of signed-graphic matroids is closed under the operations of k-sums $k \in \{1, 2, 3\}$.

Proposition 16 (Slilaty, Qin [55]). *If Σ_1 and Σ_2 are signed graphs, then $M(\Sigma_1 \oplus_k \Sigma_2) = M(\Sigma_1) \oplus_k M(\Sigma_2)$, where $k \in \{1, 2, 3\}$.*

The class of binary signed-graphic matroids is closed under k-sums for $k \in \{1, 2, 3\}$ as shown in the following theorem which appears in [55].

Theorem 25. *If $M(\Sigma_1)$ and $M(\Sigma_2)$ are both binary signed-graphic matroids, then for each $k \in \{1, 2, 3\}$, $M(\Sigma_1 \oplus_k \Sigma_2)$= $M(\Sigma_1) \oplus_k M(\Sigma_2)$ is binary.*

The link between binary signed-graphic matroids and the signed graphs representing them is the following theorem which was proved by Slilaty in [55].

Theorem 26. *If Σ is a connected signed graph, then the following are true.*

(i) If Σ is tangled, then $M(\Sigma)$ is regular.

(ii) If $M(\Sigma)$ is regular and not graphic, then Σ is tangled.

If Σ is tangled signed graph then by Theorem 3.16 in [**?**], Σ contains $-K_4$ or $\pm C_3$ as a link minor. Moreover, by Theorem 3.7 in [**?**] if Σ is connected with at least one joint and $\Sigma \backslash J_\Sigma$ is tangled then $M(\Sigma)$ is not quaternary. Tangled signed graphs were decomposed through k-sums $k \in \{1, 2, 3\}$ to balanced signed graphs and to either a projective-planar signed graph or $-K_5$ by Slilaty [54]. Many results concerning tangled signed graphs were provided also by Slilaty in [54, 45].

The class of binary signed-graphic matroids was decomposed by Papalamprou and Pitsoulis in [40]. The decomposition theorem states that a binary matroid is signed-graphic if and only if certain minors resulting from the deletion of a non-graphic cocircuit are graphic apart from exactly one which is signed-graphic. Furthermore Papalamprou and Pitsoulis provided an excluded-minor characterization of the binary signed-graphic matroids with all-graphic cocircuits and a polynomial-time algorithm recognizing whether a cographic matroid with all-graphic cocircuits is signed-graphic [41].

The class of binary signed-graphic matroids with all-graphic cocircuits was characterized by Papalamprou and Pitsoulis in [39]. Specifically they provided an excluded minor characterization for the aforementioned class of matroids which was based on the excluded minor characterization of regular signed-graphic matroids [45]. Furthermore they published a polynomial-time algorithm for determining whether a binary matroid with all-graphic cocircuits is signed-graphic.

3.6 Matroid minors

Wagner conjectured that every minor-closed class of (finite) graphs has a finite list of excluded minors. Although the conjecture for infinite graphs has failed, it has been verified for finite graphs by Robertson and Seymour's Well-Quasi-Ordering Theorem for graphs. This most celebrated result in Graph Theory says that graphs are well-quasi-ordered under the minor order. In other words, in every infinite set of graphs there is one that is isomorphic to a minor of another. Not only the theorem but also the techniques that were developed for the Graph Minors Project have numerous applications in the whole Graph Theory. Since the Graph Minors Project is matroidal in spirit, it is natural to attempt to generalize this theorem to matroids. In [35], Oxley constructs examples that show that the theorem does not extend to all matroids or even to all $\mathbb{R}$-representable matroids. However, Robertson and Seymour have conjectured that the theorem does generalize to the class of $GF(q)$-representable matroids. This conjecture known as Matroid Minor Conjecture is one of the most important open problems of Matroid Theory.

Conjecture 3.6.1 (Matroid Minor Conjecture)**.** *Let $\mathbb{F}$ be a finite field. In any finite set of $\mathbb{F}$-representable matroids there is one that is a minor of another.*

A set of graphs or a set of matroids is called an *antichain* if no member of the set is isomorphic to a minor of another member of the set. Infinite antichains exist within the class of all matroids as shown by Oxley [35], but not within the class of regular matroids. This result, namely the above conjecture for the class of regular matroids, was proved, but not published, by Seymour.

Geelen, Gerards and Whittle who work on the Matroid Minor Conjecture have published many interesting results [19, 20]. In 2008 they announced they extended Robertson's and Seymour's Graph-Minors Structure Theorem to binary matroids. The proof of Well-Quasi-Ordering Theorem for graphs is based on Graph-Minors Structure Theorem which provides a constructive characterization of the class of graphs that do not contain a given graph as a minor. Geelen, Gerards and Whittle take a similar approach in the Matroid Minors Project. In 2009 they announced that they had proved the Well-Quasi-Ordering Theorem for $q = 2$. Lately they have announced that the Matroid Minor Project has been completed.

3.7 Algorithms

Several results on matroid representation theory and matroid structure theory have important implications on combinatorial optimization and numerous computational problems concerning graphs and matroids. Based on Tutte's theorem that determines whether a binary matroid is graphic and his theory of Bridges, Bixby and Cunningham provided an algorithm which converts a linear program $\min\{cx|Ax = b, x \geq 0\}$ to a network flow problem [5]. Moreover, an algorithm that finds a separating cocircuit or a Fano minor in a binary matroid, namely a constructive proof of Tutte's corresponding theorem, was described by Cunningham in [8]. Furthermore many results on graphs have been translated to corresponding results for matroids with significant algorithmic implications. Such a result is Robertson's and Seymour's Well-Quasi-Ordering Theorem for graphs that has the following profound algorithmic consequence.

Theorem 27. *For every graph H, there is a polynomial-time, indeed an $O|V(G)|^3$, algorithm to test if a graph G has a minor isomorphic to H.*

Another direct consequence of Well-Quasi-Ordering Theorem for graphs is that for every minor closed class $\mathcal{G}$ of graphs, there is a polynomial time algorithm to test whether a graph is in $\mathcal{G}$. Robertson's and Seymour's theorem implies that the class of graphs $\mathcal{G}$ has finitely many excluded minors. Since for every graph H we can check in polynomial time whether a graph G has a minor isomorphic to H, the implementation of the above algorithm premises knowledge of the excluded minors. Therefore it cannot be practically implemented.

The last theorem has motivated an analogous conjecture for matroids. In particular, Geelen, Gerards and Whittle have conjectured that the last theorem generalizes to matroids that are representable over finite fields and, in 2009, they announced that the conjecture had been proved for binary matroids. A $GF(q)$-*represented matroid* is a matroid along with a given $GF(q)$-representation.

Theorem 28 (Minor-Recognition Conjecture)**.** *For every prime power q and every $GF(q)$-representable matroid N, there is a polynomial time algorithm for testing whether a $GF(q)$-represented matroid M has a minor isomorphic to N.*

The above conjecture has been verified when N is isomorphic to the cycle matroid of a planar graph. Furthermore, in 2003, Hliněný proved Minor-Recognition Conjecture for matroids of fixed branch-width. In contrast to the last two results, Hliněný proved that it is NP-hard to determine whether a given $\mathbb{Q}$-represented matroid of branch-width at most 3 contains a minor isomorphic to $M(C_5^+)$, where C_5^+ is the graph obtained from a cycle of length five by adding an edge that joins two non-adjacent vertices of the cycle.

Another important conjecture is the following. For every prime power q, there is a polynomial-time algorithm that, given any two matrices A_1 and A_2 over $GF(q)$ with the same set of column labels, tests whether $M[A_1] = M[A_2]$. However, Geelen published a result that contrasts this conjecture saying that it is NP-hard to test whether $M[A_1] = M[A_2]$ for two given rational matrices A_1 and A_2, even when $M[A_1]$ is a tipless free spike.

Theorem 29. *For every prime power q and every $GF(q)$-representable matroid N, there is a polynomial-time algorithm for testing whether a $GF(q)$-representable matroid M has a minor isomorphic to N.*

Chapter 4

Signed graphs

Signed graphs are well-known natural generalizations of graphs that are used to model real-world problems and to represent interactions and physical networks such as electrical networks and roadways. It is always of desire to obtain results for more general combinatorial structures than graphs such as signed graphs and matroids which arise from signed graphs, since these results have important implications to combinatorial optimization and other areas of discrete mathematics.

The main result of this thesis is the decomposition of the class of quaternary signed-graphic matroids (Theorem 53). The first steps towards the decomposition theorem was to settle the decomposition operations and to analyze structurally the class of quaternary signed-graphic matroids. In this chapter, we define a new operation on signed graphs, the star composition, that is used for the decomposition of quaternary signed-graphic matroids and we determine the structural properties of signed graphs that represent quaternary signed-graphic matroids. Specifically we obtain structural results for signed graphs which are cylindrical or have a balancing vertex up to deleting any joints. Since cylindrical signed graphs have a planar embedding by definition, it is natural to start with investigating the structural properties of planar signed graphs. The structural results that are obtained for the above signed graphs are translated to structural results for the class of quaternary signed-graphic matroids.

4.1 Decomposition operations for signed graphs

The decomposition of quaternary signed-graphic matroids is based not only on matroidal operations but also on signed graphic operations. An operation that is used to decompose a matroid to some well-defined minors induces naturally an operation that composes it from its building blocks. In the following, we define a new operation for signed graphs called star composition while the reverse operation is called star decomposition. Moreover, we present Slilaty's definition of k-sum $k \in \{1, 2, 3\}$ of signed graphs since it is used extensively.

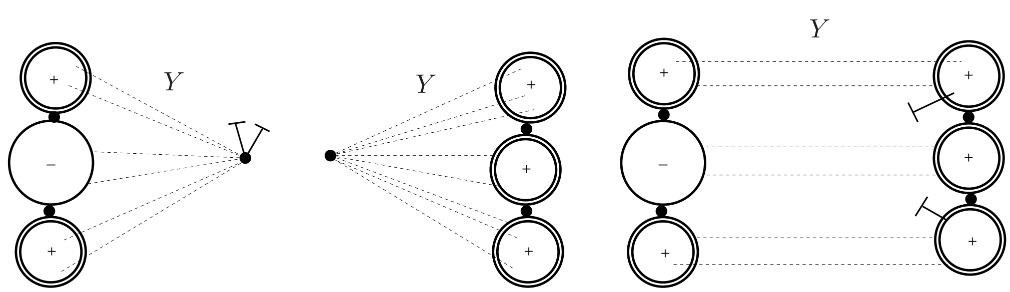
4.1.1 Star decomposition

The star composition of two graphs G_1 and G_2 in Y is defined in terms of incidence matrices in [44]. Generalizing this operations for signed graphs, we define the operation *star composition* of two signed graphs with respect to Y. Let $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ be two signed graphs such that Σ_2 is balanced. Suppose further that Y is the star of a vertex in both Σ_1 and Σ_2. The *star composition* of Σ_1 and Σ_2 with respect to Y is the signed graph $\Sigma = (G, \sigma)$ such that the underlying graph G is obtained from the graphs $G_1 \backslash Y$ and $G_2 \backslash Y$ as follows:

(a) by adding a link between the end-vertex of the link of Y in G_1 and the end-vertex of the identical link of Y in G_2 or

(b) by adding a joint at the end-vertex of the link of Y in G_2 when the identical element of Y in G_1 corresponds to a joint.

The sign of an edge in Σ is the sign which is attributed to the edge by σ_1 when it belongs to G_1 and the sign which is attributed to the edge by σ_2 if it belongs to $G_2 \backslash Y$.



(a) The signed graph Σ_1 (b) The signed graph Σ_2 (c) The signed graph Σ

Figure 4.1: The star composition of Σ_1 and Σ_2

The reverse operation of star composition of two signed graphs is called *star decomposition* of a signed graph with respect to Y. Given an unbalancing bond Y of a signed graph Σ, the star decomposition of Σ with respect to Y results into two signed graphs Σ_1 and Σ_2 where Y is the star of a vertex in both signed graphs.

4.1.2 k-sums for signed graphs

The definition of k-sum $k \in \{1, 2, 3\}$ for two signed graphs is given by Slilaty in [55]. The k-*sum*, $k \in \{1, 2, 3\}$ of two signed graphs Σ_1 and Σ_2, denoted by $\Sigma_1 \oplus_k \Sigma_2$, induces naturally a decomposition operation. In the following we present the definitions of the decomposition of the signed graph $\Sigma = \Sigma_1 \oplus_k \Sigma_2$ to Σ_1 and Σ_2.

1-sum

1-*sum* of two signed graphs Σ_1 and Σ_2, where one of them is unbalanced, is the identification of Σ_1 and Σ_2 at a vertex v. Accordingly $\Sigma = \Sigma_1 \oplus_1 \Sigma_2$ is decomposed to the signed graphs Σ_1 and Σ_2 by splitting vertex v to two vertices v_1 and v_2.

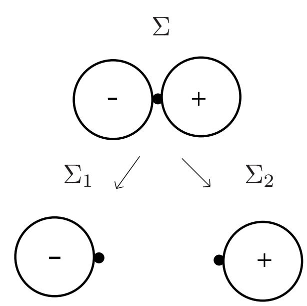


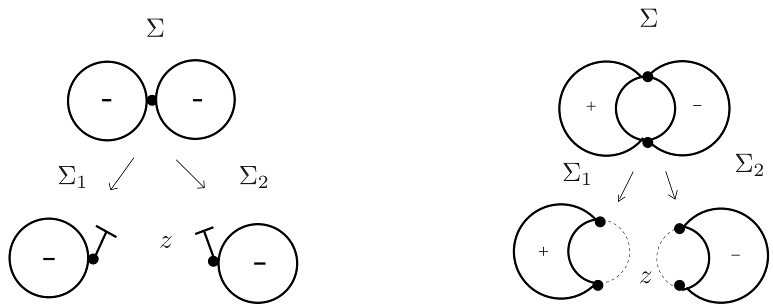
Figure 4.2: $\Sigma = \Sigma_1 \oplus_1 \Sigma_2$

2-sum

The 2-*sum* of two signed graphs is taken along an edge, that is not a coloop in each of the corresponding signed-graphic matroids. Moreover, it is distinguished in 1-*vertex* 2-*sum* and 2-*vertex* 2-*sum*. If Σ_1 and Σ_2 are unbalanced signed graphs, then the 1-vertex 2-sum of Σ_1 and Σ_2 is obtained by identifying them along a joint and then deleting the joint. If exactly one of Σ_1, Σ_2 is unbalanced then the 2-vertex 2-sum of Σ_1 and Σ_2 is obtained by identifying them at a link of the same sign, applying switchings if necessary, and then deleting the link.

3-sum

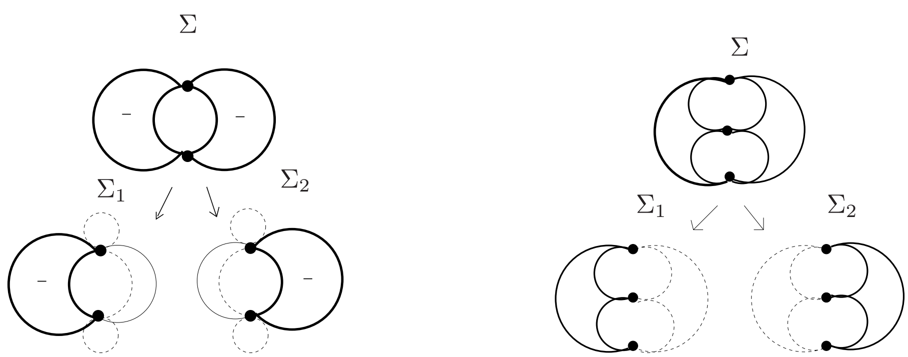
The 3-*sum* of two signed graphs is defined when the signed-graphic matroid of each of them has rank at least three. The 3-*sum* has the following two types 2-*vertex* 3-*sum* and 3-*vertex* 3-*sum*. If Σ_1 and Σ_2 are both unbalanced signed graphs then the 2-vertex 3-sum of Σ_1 and Σ_2 is obtained by identifying them along a 4-edge



(a) Σ is 1-vertex 2-sum of Σ_1 and Σ_2 (b) Σ is 2-vertex 2-sum of Σ_1 and Σ_2

Figure 4.3: $\Sigma = \Sigma_1 \oplus_2 \Sigma_2$.

line in each (see Figure 3.1) and then deleting the edges of the line. If exactly one of Σ_1 and Σ_2 is unbalanced, then the 3-vertex 3-sum of Σ_1 and Σ_2 is obtained by identifying them along a triangle so that the corresponding links have the same sign, applying switchings if necessary, and then deleting the edges of the triangle.



(a) Σ is 2-vertex 3-sum of Σ_1 and Σ_2 (b) Σ is 3-vertex 3-sum of Σ_1 and Σ_2

Figure 4.4: $\Sigma = \Sigma_1 \oplus_3 \Sigma_2$.

Given two balanced signed graphs $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$, the k-sum of Σ_1 and Σ_2 is the signed graph $\Sigma = (G, \sigma)$ with underlying graph $G = G_1 \oplus_k G_2$ and $\sigma(e) = +1$, for every edge $e \in E(\Sigma)$.

4.2 Planar signed graphs

In this subsection, we furnish structural results for planar signed graphs concerning negative cycles and negative faces. Initially we give some definitions which are used in the following.

A signed graph is *planar* if and only if its underlying graph is planar. Let Σ be a planar signed graph such that $\Sigma\backslash J_{\Sigma}$ is 2-connected. We define the faces of Σ to be the faces of $\Sigma\backslash J_{\Sigma}$. Moreover, we define the faces of $\Sigma\backslash J_{\Sigma}$ to be the faces of its underlying graph, while the *sign of a face* to be the sign of the cycle which defines its boundary. Two faces are *incident* if they share at least one edge and *vertex-disjoint* if they have no common vertex. Let H be a subgraph of Σ. If the boundary of an inner face F of Σ belongs to H then we say that F is an inner face of H. We note that every face of Σ is bounded by a cycle and that every inner face of H is an inner face of Σ. If F is an inner face of H and H has a cycle boundary C then we say that F is *contained in the cycle* C or it is in C. Since Σ has a planar embedding with F being the outer face, it is straighforward to see that every cylindrical signed graph, which has a planar embedding with a negative face, has also a planar embedding with a negative outer face. Henceforth for connected planar signed graphs with a negative face we always assume a planar embedding with a negative outer face.

In a 2-connected planar graph G, there exists an inner face that has a non-empty intersection with the outer cycle of G and, moreover, the elements of this intersection induce a path. The above can be easily generalized for a planar graph G' with half-edges and loops such that $G'\backslash J_{G'}$ is 2-connected.

Lemma 4.2.1. *Let C be the outer cycle of a planar graph G such that $G\backslash J_G$ is 2-connected then there exists an inner face F such that $E(C) \cap E(F) \neq \emptyset$ and $G[E(C) \cap E(F)]$ is a path.*

Proof. Let C be the outer cycle of G. Since any edge of C is adjacent to exactly two faces (see Lemma 4.2.2 in [10]) there exists an inner face F' of G such that $E(F') \cap E(C) \neq \emptyset$. Assume that $G[E(F) \cap E(C)]$ is not a path; since, otherwise, the result follows. Therefore, there exist vertex-disjoint paths $K_1, K_2, \ldots, K_m$ with $m \geq 2$ whose edges belong in $E(F') \cap E(C)$. Since G is a 2-connected planar graph, the boundary of F' is a cycle of G (see Proposition 4.2.6 in [10]). Thus, there exist distinct paths $P_1, P_2, \ldots, P_n$ with $n \geq 2$ of the boundary of F' that are internally disjoint from C but have both of their end-vertices at $V(C)$. As a result, the edges of any P_i ($i = 1, \ldots, n$) and the edges of C induce a theta graph. In that theta graph, let H_i be the path of C that has the same end-vertices with P_i and is internally vertex disjoint with the boundary of F. Let also G_i be the subgraph of G that is bounded by the cycle $P_i \cup H_i$. Since the end-vertices of H_i are vertices of some K_l and K_m (with $l \neq m$) and F' is a face, K_l and K_m are the only paths that connect G_i with $G\backslash(K_l \cup G_i \cup K_m)$. If we delete $E(F') \cap E(C)$ from G, each G_i is a connected component in the resulting disconnected graph whose faces form a subset of the set of faces of G and furthermore, for each G_i, $E(G_i) \cap E(C) \neq \emptyset$ induces a path. We select arbitrarily a G_i. Clearly, graph G_i has as outer cycle the $P_i \cup H_i$ and there exists either a face F'' such that $G[E(C) \cap E(F'')]$ is either a path and the result follows or a set of disconnected paths. In the latter case we delete these disconnected paths as we did in the case of G and a set of connected components is obtained. This procedure is iteratively applied until either the intersection of a face of a component obtained from deletion of disconnected paths and C is a path or the component consists of a single face in which case also the result follows. □

Utilising the aforementioned result the following lemma is proved, which illustrates the relationship between negative faces and negative cycles in a planar signed graph.

Lemma 4.2.2. *In a planar signed graph Σ such that $\Sigma \backslash J_\Sigma$ is 2-connected, every negative cycle contains a negative face.*

Proof. Let C be a negative cycle in a planar signed graph. By Lemma 4.2.1, there exists a face F in C such that their common edges form a path. Let $P = E(C) \cap E(F)$, $H = E(F) \backslash E(C)$ and $K = E(C) \backslash E(F)$. If F is a negative face then there is nothing to prove. In the remaining case, we assume that the boundary of F is a positive cycle in Σ. In the theta graph induced by $P \cup K \cup H$, the cycle induced by $K \cup H$ is a negative one, since in signed graphs any theta subgraph contains 0 or 2 negative cycles. Therefore, iteratively we come across either a negative face or a negative cycle and since the number of faces is finite the result follows. □

As proved in the following result, the number of negative faces in a negative cycle is odd. As a direct consequence every positive cycle contains an even number of negative faces.

Lemma 4.2.3. *In a planar signed graph Σ such that $\Sigma \backslash J_\Sigma$ is 2-connected, every negative cycle contains an odd number of negative faces whereas every positive cycle contains an even number.*

Proof. Induction on the number of faces contained in a cycle C of a planar signed graph shall be applied. If C contains one face, i.e. it is a face itself, the statement holds. For the induction hypothesis, assume that it holds for every C which has fewer faces than n. It will be shown that the statement holds for every C with n faces. Since the signed graph is 2-connected, without loss of generality we can assume that C is the outer cycle. Since C contains $n > 1$ faces there is a path internally disjoint from C with both end-vertices belonging to $V(C)$. Thus, a theta graph is formed by C and that path. If we assume that C is negative, then in that theta graph, the chord P divides C into a positive cycle C^+ and a negative cycle

C^- due to the fact that a theta graph in a signed graph contains either 0 or 2 negative cycles. By the induction hypothesis, C^+ has an even number of negative faces, while C^- has an odd number. Since the set of faces contained in C^+ is disjoint from the set of faces contained in C^- while their union is the complete set of faces contained in C, it follows that C contains an odd number of negative faces. In the remaining case, i.e. if we assume that C is a positive cycle, then C is divided into either two positive or two negative cycles. Therefore, similarly, using the induction hypothesis, C is shown to have an even number of negative faces. □

The following corollary is easily obtained from Lemma 4.2.3 by taking into account the sign of the outer face of a 2-connected planar signed graph.

Corollary 3. *The number of negative faces in a planar signed graph Σ such that $\Sigma \backslash J_\Sigma$ is 2-connected is even.*

4.3 Cylindrical signed graphs with joints

An important class of planar signed graphs is the class of cylindrical signed graphs, since they constitute representations of quaternary signed-graphic matroids (Theorem 24). Structural properties of signed graphs which are cylindrical or have a balancing vertex after removing any joints enable us to derive results for the associated quaternary signed-graphic matroids.

For 2-connected cylindrical signed graphs, the existence of vertex-disjoint negative faces determines whether the associated matroid is graphic or not, as shown in the following result.

Theorem 30. *Let Σ be a 2-connected cylindrical signed graph. $M(\Sigma)$ is graphic if and only if Σ has no two vertex-disjoint negative faces.*

Proof. For necessity assume on the contrary that Σ has two vertex-disjoint negative faces. Then there are two vertex-disjoint negative cycles in Σ and $M(\Sigma)$ is non-binary as indicated by Theorem 22. For sufficiency, if we assume that Σ has no negative faces then Σ does not have a negative cycle due to Lemma 4.2.2; therefore, Σ is balanced and, by Proposition 12, $M(\Sigma)$ is graphic. Now if we assume that Σ has one negative face then by Corollary 3, Σ is not 2-connected. Since Σ is cylindrical, i.e. it has at most two negative faces, the case left to be considered is when Σ has exactly two negative faces, say C_1 and C_2, which, by hypothesis, are not vertex-disjoint. Let $V = V(C_1) \cap V(C_2)$ and consider a negative cycle C of Σ, where C is other than the cycles defined by the boundaries of C_1 and C_2. By Lemma 4.2.3, C contains exactly one from C_1 and C_2. This fact along with planarity of Σ implies that $V \subseteq V(C)$. Therefore, the vertices in V are also vertices of any negative cycle of Σ. This implies that any vertex in V is a balancing vertex of Σ and thus, by Proposition 12, $M(\Sigma)$ is graphic. $\square$

When a signed graph $\Sigma \backslash J_\Sigma$ is cylindrical, by definition it has a planar embedding with at most two negative faces. Since the addition of joints does not affect the number of negative faces, Σ has a planar embedding with at most two negative faces. Moreover, by Corollary 3, we have that the number of negative faces in a vertically 2-connected cylindrical signed graph is even.

Regarding a signed graph Σ such that $\Sigma \backslash J_\Sigma$ is cylindrical or has a balancing vertex, the following two technical results are also proved.

Lemma 4.3.1. *Let Σ be a vertically 2-connected cylindrical signed graph such that $M(\Sigma)$ is quaternary and non-binary. If Y is a non-graphic cocircuit of $M(\Sigma)$, then for each separator B of an unbalanced component of $\Sigma \backslash Y$ there exists at most one vertex of attachment $v \in V(B)$ with balanced $C(B,v)$ such that $Y(B,v)$ consists of edges of different sign.*

Proof. Let us assume first that Σ is jointless. Since $M(\Sigma)$ is non-binary, by Theorem 30, Σ has two vertex-disjoint negative faces. Moreover, Y is a non-balancing bond and, therefore, in $\Sigma \backslash Y$ there exist a balanced and an unbalanced component, denoted by Σ^+ and Σ^-, respectively. By performing switchings at vertices of Σ, all edges of the balanced separators of $\Sigma \backslash Y$ can become positive. Thus, in what follows, we can assume that only Y and the unbalanced separators of $\Sigma \backslash Y$ may contain edges of negative sign.

By way of contradiction, let v_1 and v_2 be two vertices of attachment of a bridge B of Y in Σ^- such that each $C(B,v_i)$ is balanced and each $Y(B,v_i)$ consists of edges of different sign (where $i = 1,2$). Clearly each $C(B,v_i)$ is incident with at least two edges of Y. Let us denote by $y_i = \{u_i, w_i\}$ and by $y'_i = \{u'_i, w'_i\}$ two edges of different sign in $Y(B,v_i)$, where u_i and u'_i are vertices in Σ^- and w_i and w'_i are vertices in Σ^+. Due to the fact that $C(B,v_i)$ is connected and balanced, there exists a positive path P_i between u_i and u'_i in Σ^- while, due to the fact that Σ^+ is connected and balanced, there exists a positive path P'_i between w_i and w'_i in Σ^+. Therefore, the cycle C_i formed by P_i, P'_i, y_i and y_i is of negative sign. Hence, by Lemma 4.2.2, a negative face F_i is contained in each C_i. Moreover, since Y is a non-graphic cocircuit, there would be at least one non-graphic bridge of Y in $M(\Sigma)$; otherwise, $M(\Sigma) \backslash Y$ would be graphic; by Proposition 12, the corresponding unbalanced separator should contain a negative cycle other than joint. Thus, by Lemma 4.2.2, a negative face is contained in that separator. Clearly, this negative face is distinct from F_1 and F_2, since they have different boundaries. This means that Σ has three distinct negative faces which is in contradiction with the hypothesis saying that $\Sigma\backslash Y$ is cylindrical. If Σ has joints and two negative faces, then the result follows as above. $\square$

Lemma 4.3.2. *Let $M(\Sigma)$ be a connected quaternary non-binary signed-graphic matroid such that $\Sigma\backslash J_\Sigma$ has a balancing vertex. If Y is a non-graphic cocircuit of $M(\Sigma)$, then for each separator B of an unbalanced component of $\Sigma\backslash Y$ there exists at most one vertex of attachment $v \in V(B)$ with balanced $C(B, v)$ such that $Y(B, v)$ consists of edges of different sign.*

Proof. Since Y is non-balancing, $\Sigma\backslash Y$ consists of a balanced component denoted by Σ^+ and some unbalanced components. Perform switchings at vertices of Σ such that all edges of the balanced separators of $\Sigma\backslash Y$ become positive. Therefore, in what follows, we can assume that only Y and the unbalanced separators of $\Sigma\backslash Y$ may contain edges of negative sign.

By way of contradiction, assume that B is a separator of an unbalanced component of $\Sigma\backslash Y$ which has two vertices of attachment $v_i, (i = 1, 2)$ with balanced $C(B, v_i)$ such that $Y(B, v_i)$ consists of edges of different sign. Clearly each $C(B, v_i)$ is incident with at least two edges of Y. Let us denote by $y_i = \{u_i, w_i\}$ and by $y'_i = \{u'_i, w'_i\}$ the two edges of different sign in $Y(B, v_i)$, where u_i and u'_i are vertices in Σ^- and w_i and w'_i are vertices in Σ^+. Due to the fact that $C(B, v_i)$ is connected and balanced, there exists a positive path P_i between u_i and u'_i in Σ^- while, due to the fact that Σ^+ is connected and balanced, there exists a positive path P'_i between w_i and w'_i in Σ^+. Therefore, the cycle C_i formed by P_i, P'_i, y_i and y'_i is of negative sign. Since $\Sigma\backslash J_\Sigma$ has a balancing vertex, C_i are not vertex-disjoint. By the definition of $C(B, v_i)$ and the fact that v_1 and v_2 are distinct, these cycles may share vertices belonging only in Σ^+, and, thus,the balancing vertex of $\Sigma\backslash J_\Sigma$ should be a vertex of Σ^+. However, there is a negative cycle in Σ^- which contradicts the fact that there is a balancing vertex in Σ^+. $\square$

Let M be a matroid, if C is a circuit and C^* is a cocircuit, then $|C \cap C^*| \neq 1$. This property is called *orthogonality*. The next lemma derives easily from the orthogonality property of matroids.

Lemma 4.3.3. *The boundary of a face in a planar embedding of a 2-connected cylindrical signed graph Σ contains zero or two edges of an unbalancing bond Y in Σ.*

Proof. Since Σ is cylindrical and Y is an unbalancing bond in Σ then $\Sigma\backslash Y$ consists of one unbalanced and one balanced connected component denoted by Σ_1 and Σ_2, respectively. It is well-known that any cycle of a graph intersects any bond in an even number of edges. Therefore, since a face may also be viewed as a cycle (see Proposition 4.2.6 in [10]), it remains to show that the boundary of any face can not contain more than two edges of Y. By way of contradiction, assume that the boundary of a face F in Σ has more than two common edges with Y. Let us traverse F starting from an edge y_1 of Y with endpoint v_1 in Σ_1 while let us call y_2 the next edge of Y that we encounter in that traversal. Let also v_2 be the endpoint of y_2 in Σ_1. By the fact that for any two points of the plane lying in F there exists a simple curve joining them (without crossing any edge), we can say that there is no path connecting v_1 and v_2 in Σ_1; a contradiction, since Σ_1 is connected. $\square$

Chapter 5

Signed-graphic matroids

A key issue in many problems of matroid representation theory and matroid structure theory is to determine the structural properties of the class of matroids under examination. In this chapter, we provide structural results for the class of signed-graphic matroids, in view of the decomposition theorem for quaternary signed-graphic matroids (Theorem 53). More precisely, we characterize graphically matroidal notions and we present structural results for cocircuits and bonds. Furthermore we determine hereditary properties of cocircuits through k-sums. The structural results obtained help us to establish necessary and sufficient conditions for a quaternary matroid to be signed-graphic. Cocircuits and bonds play a central role in the decomposition of the classes of graphic matroids and binary signed-graphic matroids, since the decomposition operation is the deletion of a cocircuit. Cocircuits have also a crucial role in the decomposition of quaternary signed-graphic matroids. Hence we investigate properties of cocircuits and their bridges such as bridge-separability and avoidance.

The concepts of bridge-separability and avoidance, which were defined in [61], were used by Tutte for the decomposition of graphic matroids. The same concepts were also used by Papalamprou and Pitsoulis in [40] for the decomposition of binary signed-graphic matroids. Avoidance and bridge-separability prove to be useful tools in order to extend the latter decomposition theorem to quaternary signed-graphic matroids. Proving that these properties are preserved under the operation of k-sums of matroids $k \in \{1, 2, 3\}$, we obtain the desirable graphical representations for the highly connected minors of quaternary signed-graphic matroids. Following the matroidal definition of avoidance, we shall say that two separators of a bond Y in Σ are avoiding when the corresponding bridges of Y in $M(\Sigma)$ are avoiding.

5.1 Cocircuits and bonds

Cocircuits are of basic importance for the decomposition of quaternary signed-graphic matroids, since the deletion of a non-graphic cocircuit is the decomposition operation. The following result guarantees the existence of a non-graphic and bridge-separable cocircuit in a signed-graphic matroid with not all-graphic cocircuits.

Lemma 5.1.1. *If a connected signed-graphic matroid $M(\Sigma)$ has a non-graphic cocircuit which corresponds to a double bond in Σ whose balancing part contains links, then $M(\Sigma)$ has also a non-graphic cocircuit which corresponds to an unbalancing bond or a double bond in Σ whose balancing part contains only joints.*

Proof. Let us denote with Y the non-graphic cocircuit of $M(\Sigma)$ which is a double bond whose balancing part contains links in Σ. Then $\Sigma\backslash Y$ consists of one balanced component, denoted by Σ^+ and some unbalanced components. Since Y is non-graphic, there are unbalanced separators in $\Sigma\backslash Y$ that are non-graphic bridges of Y in $M(\Sigma)$. We perform switchings in $V(\Sigma^+)$ so that the edges of the balancing part of Y become negative. Consider an edge $e = \{v_1, v_2\}$ of the balancing part of Y and the partition $(\{v_1, v_2\}, V(\Sigma)\backslash\{v_1, v_2\})$ of $V(\Sigma)$. We distinguish the following two cases:
<u>Case 1:</u> The signed graph $\Sigma[V(\Sigma)\backslash\{v_1, v_2\}]$ is connected.
Since Y is a double bond of Σ and $v_1, v_2 \in V(\Sigma^+)$, the signed graph $\Sigma[V(\Sigma)\backslash\{v_1, v_2\}]$ contains the unbalanced component of $\Sigma\backslash Y$ as a subgraph. Then the signed graph $\Sigma\backslash star(v_i)$ where $v_i \in \{v_1, v_2\}$ consists of a balanced component, that is the vertex v_i, and an unbalanced component induced by $V(\Sigma)\backslash\{v_i\}$. By definition all edges of $star(v_i)$ have one end-vertex at the balanced component of $\Sigma\backslash star(v_i)$ and one at $\Sigma[V(\Sigma)\backslash\{v_i\}]$ or the unique end-vertex at the balanced component of $\Sigma\backslash star(v_i)$. Furthermore, $star(v_i)$ is minimal with respect to increasing the number of balanced components and, therefore, it is either an unbalancing bond or a double bond whose balancing part contains only joints in Σ. By the fact that $\Sigma[V(\Sigma)\backslash\{v_1, v_2\}]$ is a subgraph of $\Sigma[V(\Sigma)\backslash\{v_i\}]$, the latter component of $\Sigma\backslash star(v_i)$ contains the unbalanced separator of $\Sigma\backslash Y$ that is a non-graphic bridge of Y in $M(\Sigma)$. Thus, the cocircuit $star(v_i)$ of $M(\Sigma)$ is non-graphic.
<u>Case 2:</u> The signed graph $\Sigma[V(\Sigma)\backslash\{v_1, v_2\}]$ is disconnected.
Let $S_1, \ldots, S_m$ denote the connected components of $\Sigma[V(\Sigma)\backslash\{v_1, v_2\}]$. If each S_l, $l = 1, \ldots, m$ is unbalanced, then $\Sigma\backslash star(v_i)$ where $v_i \in \{v_1, v_2\}$ consists of one balanced component, that is the vertex v_i, and the unbalanced connected components of $\Sigma[V(\Sigma)\backslash\{v_i\}]$. Hence $star(v_i)$ is either an unbalancing bond or a double bond whose balancing part contains only joints in Σ. Moreover, since Y is a non-graphic

cocircuit of $M(\Sigma)$, the cocircuit $star(v_i)$ is non-graphic. Otherwise there is a balanced component $S_k, k \in \{1, \ldots, m\}$. We shall denote with H the set of edges of $E(\Sigma)$ that have one end-vertex in $\{v_1, v_2\}$ and one in $V(\Sigma)\backslash\{v_1, v_2\}$ (see Figure 5.1). For each $v_i \in \{v_1, v_2\}$ and $v' \in V(S_k)$, each $v_i v'$-path contains an edge of H. Let H' be the proper subset of H whose one end-vertex belongs to $V(S_k)$ and one to $\{v_1, v_2\}$ and let J_k be the (possibly empty) set of joints with an end-vertex at $V(S_k)$ (in the example signed graph of Figure 5.1, the links of Y are illustrated with solid lines while the edges of $H' \cup J_k$ are illustrated with dashed lines). By the fact that Σ is connected, the signed graph $\Sigma\backslash(H' \cup J_k)$ consists of one balanced connected component S_k and one unbalanced connected component which is the union of all $S_l, (l \neq k)$, the edge e and the edges $H\backslash H'$. Therefore $H' \cup J_k$ is a minimal set of edges consisting of links with one end-vertex at S_k and one at the unique unbalanced component of $\Sigma\backslash(H' \cup J_k)$ or joints attached at S_k. This implies that $H' \cup J_k$ constitutes a non-graphic cocircuit corresponding to either an unbalancing bond or a double bond whose balancing part contains only joints in Σ. $\square$

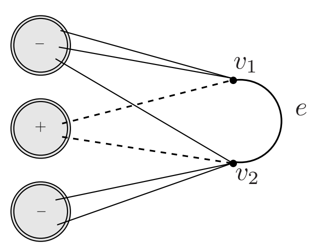


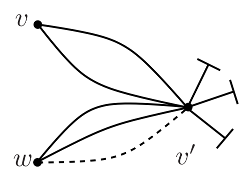
Figure 5.1: A double bond whose balancing part contains links

The cocircuit Y of a signed-graphic matroid $M(\Sigma)$ remains a cocircuit in the minor $M(\Sigma).(B \cup Y)|Y$ by the definitions of matroidal contraction and deletion. In the following lemma, the sets of elements of Y which correspond to bonds in the signed graph $\Sigma.(B \cup Y)|Y$ are characterized graphically. Moreover, Lemma 5.1.2 is used for the graphical characterization of the sets of $\pi(M, B, Y)$ for a matroid M and a bridge B of a cocircuit Y in M in Lemma 5.1.3.

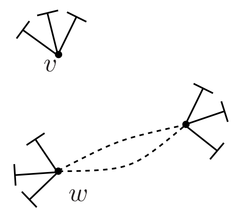
Lemma 5.1.2. *Let Y be a cocircuit of a connected signed-graphic matroid $M(\Sigma)$ and B a separator of $\Sigma\backslash Y$. A bond of $\Sigma.(B \cup Y)|Y$ is either:*

(i) the star of a vertex,

(ii) a maximal set of parallel links of the same sign with the joints, at one end-vertex



(a) $B \in \Sigma^-$



(b) $B \in \Sigma^+$

Figure 5.2: The signed graph $\Sigma.(B \cup Y)|Y$.

(iii) the set of joints at a vertex.

Proof. Let us suppose first that Y is a non-balancing bond in Σ. Then the signed graph $\Sigma\backslash Y$ consists of exactly one balanced component Σ^+ and one or more unbalanced components. Let us assume that B is an unbalanced separator of an unbalanced component in $\Sigma\backslash Y$ denoted by Σ^-. The signed graph $\Sigma.(B \cup Y)|Y$ is obtained from Σ by contracting Σ^+, each unbalanced component of $\Sigma\backslash Y$ different from Σ^-, each $C(B, v)$ where $v \in V(B)$ and finally by deleting B. Thus, $\Sigma.(B \cup Y)|Y$ consists only of edges of Y which are either parallel links incident at a vertex of attachment of B or joints at the vertex v' where Σ^+ is contracted (see Figure 5.2(a)). By definition, a bond in $\Sigma.(B \cup Y)|Y$ is either the star of a vertex or a maximal set of parallel links of the same sign incident at a vertex of attachment of B with the joints at their other end-vertex or the set of joints at v'. The case is similar when B is a balanced separator of an unbalanced component in $\Sigma\backslash Y$.

Let us assume that B is a balanced separator of Σ^+. The balancing set of Y consists of links or joints that have both end-vertices or one end-vertex at Σ^+ respectively, while the unbalancing set of Y consists of links which have one end-vertex at Σ^+ and one at an unbalanced component of $\Sigma\backslash Y$ by minimality of Y. To obtain the signed graph $\Sigma.(B \cup Y)|Y$, the unbalanced components of $\Sigma\backslash Y$ are contracted and therefore the edges of the unbalancing set of Y become half-edges at their other end-vertex in Σ^+. Hence the signed graph $\Sigma.(B \cup Y)|Y$ consists only of edges of Y which are either half-edges at a vertex of attachment of B or parallel links of the same sign incident at two vertices of attachment of B (see Figure 5.2(b)). By definition it follows that a bond in $\Sigma.(B \cup Y)|Y$ is either the star of a vertex or a maximal set of parallel links of the same sign with the joints at one end-vertex or the set of joints at a vertex of attachment of B.

Let us suppose now that Y is a balancing bond in Σ. Then the signed graph $\Sigma\backslash Y$ consists of one balanced component denoted also by Σ^+. The edges of Y are either half-edges at a vertex of Σ^+ or parallel links of the same sign incident at two vertices of Σ^+. Each separator of Σ^+ is balanced, therefore the signed graph $\Sigma.(B \cup Y)|Y$ consists of half-edges at a vertex of attachment of B or parallel links of the same sign incident at two vertices of attachment of B (see Figure 5.2(b)). Thus by definition a bond in $\Sigma.(B \cup Y)|Y$ is either (i), (ii) or (iii). □

Let Y be a cocircuit of a matroid M and let B be a bridge of Y in M, the following result is a straightforward consequence of the definition of $\pi(M, B, Y)$ and Lemma 5.1.2.

Lemma 5.1.3. *Let Y be a cocircuit of a signed-graphic matroid $M(\Sigma)$ and B a separator of $\Sigma\backslash Y$. The members of $\pi(M(\Sigma), B, Y)$ are sets of edges in $\Sigma.(B \cup Y)|Y$ which can be either:*

(i) maximal set of parallel edges of the same sign, or

(ii) the set of joints at a vertex.

Jointless signed graphs arise in the k-sum decomposition of quaternary signed-graphic matroids by Theorem 24. In the following propositions, we characterize graphically cocircuits and we determine properties of bonds of signed graphs that result after removing any joints.

Given a bond Y in a signed graph Σ, there exists a subset of Y that is a bond in the signed graph $\Sigma\backslash J_\Sigma$ preserving the type of Y.

Proposition 17. *If Y is a non-graphic cocircuit of $M(\Sigma)$, then there exists a cocircuit $Y' \subseteq Y\backslash J_\Sigma$ which is a non-balancing bond in $\Sigma\backslash J_\Sigma$.*

Proof. Since Y is a non-graphic cocircuit of $M(\Sigma)$, it corresponds to a non-balancing bond in Σ. Moreover, there is an unbalanced separator in $\Sigma\backslash Y$ that contains a negative cycle C^- which is not a joint. By the definitions of matroidal deletion and that of cocircuits of a matroid, Y' is a cocircuit of $M(\Sigma\backslash J_\Sigma)$. Since $Y' \subseteq Y$, the signed graph $(\Sigma\backslash J_\Sigma)\backslash Y'$ has an unbalanced separator that contains C^-. Due to the fact that Y' is a cocircuit of $M(\Sigma\backslash J_\Sigma)$, it follows that Y' is a non-balancing bond of $\Sigma\backslash J_\Sigma$. □

The following result can be proven in a similar fashion.

Proposition 18. *If Y is a graphic cocircuit of $M(\Sigma)$, then there exists $Y' \subseteq Y\backslash J_\Sigma$ which is a graphic cocircuit of $M(\Sigma\backslash J_\Sigma)$.*

The following two results are easily obtained combining the above propositions and the definitions of a star and a balancing bond in a signed graph.

Proposition 19. *If $Y \in \mathcal{C}^*(M(\Sigma))$ and $Y' \subseteq Y \backslash J_\Sigma$ is a cocircuit of $M(\Sigma \backslash J_\Sigma)$ and the star of a vertex in $\Sigma \backslash J_\Sigma$, then Y is the star of a vertex in Σ.*

Proposition 20. *If $Y \in \mathcal{C}^*(M(\Sigma))$ and $Y' \subseteq Y \backslash J_\Sigma$ is a cocircuit of $M(\Sigma \backslash J_\Sigma)$ and balancing bond in $\Sigma \backslash J_\Sigma$, then Y is a balancing bond in Σ.*

The next two technical lemmas are used to prove Theorem 31 which is essential to the decomposition of quaternary signed-graphic matroids.

Lemma 5.1.4. *If Y is a non-graphic cocircuit of a connected signed-graphic matroid $M(\Sigma)$ and $Y \backslash J_\Sigma$ is a cocircuit of $M(\Sigma \backslash J_\Sigma)$, then there is no joint unbalanced component in $\Sigma \backslash Y$.*

Proof. By hypothesis Y corresponds to a non-balancing bond in Σ, therefore the signed graph $\Sigma \backslash Y$ has at least one unbalanced connected component. By way of contradiction, let us assume that there is a joint unbalanced component in $\Sigma \backslash Y$, denoted by Σ_J^- and therefore the component $\Sigma_J^- \backslash J_\Sigma$ is balanced in the signed graph $\Sigma \backslash J_\Sigma$. Since Y is a non-graphic cocircuit, there is an unbalanced component in $\Sigma \backslash Y$, other than Σ_J^-, that contains an unbalanced separator corresponding to a non-graphic bridge. If the edges of the unbalancing part of $Y \backslash J_\Sigma$ that have an end-vertex at $\Sigma_J^- \backslash J_\Sigma$ have different sign, then they constitute a minimal set whose deletion from $\Sigma \backslash J_\Sigma$ increases the number of balanced components. Since this minimal set of edges is contained properly in $Y \backslash J_\Sigma$, it follows that $Y \backslash J_\Sigma$ is not a bond in $\Sigma \backslash J_\Sigma$. Otherwise the edges of the unbalancing part of $Y \backslash J_\Sigma$ that have an end-vertex at $\Sigma_J^- \backslash J_\Sigma$ have the same sign. The component which is formed by $\Sigma_J^- \backslash J_\Sigma$, the edges of the unbalancing part of $Y \backslash J_\Sigma$ that have an end-vertex at $\Sigma_J^- \backslash J_\Sigma$ and the balanced component of $\Sigma \backslash Y$ is balanced. Moreover, the set of edges of the unbalancing part of $Y \backslash J_\Sigma$ that have an end-vertex at an unbalanced component of $\Sigma \backslash Y$ other than Σ_J^- is minimal with respect to increasing the number of balanced components. Thus it is a bond of $\Sigma \backslash J_\Sigma$ and proper subset of $Y \backslash J_\Sigma$ which is a contradiction. □

Lemma 5.1.5. *Let B be a bridge of a cocircuit Y of a signed-graphic matroid $M(\Sigma)$ and B' be a bridge of a cocircuit $Y' \subseteq Y \backslash J_\Sigma$ of the signed-graphic matroid $M(\Sigma \backslash J_\Sigma)$. Suppose further that Y is a non-balancing bond in Σ and $S \in \pi(M(\Sigma), B, Y)$. Then one of the following holds:*

(i) If $Y' = Y \backslash J_\Sigma$, then $B' \subseteq B \backslash J_\Sigma$ and there exists $S' \in \pi(M(\Sigma \backslash J_\Sigma), B', Y')$ such that $S \backslash J_\Sigma \subseteq S'$.

(ii) If $Y' \subset Y \backslash J_\Sigma$, then there exists $S' \in \pi(M(\Sigma \backslash J_\Sigma), B', Y')$ such that $S' \subseteq S \backslash J_\Sigma$.

Proof. We distinguish the following two cases:
Case 1: $Y' = Y \backslash J_\Sigma$
By Lemma 5.1.4, there is no joint unbalanced component in $\Sigma \backslash Y$ and therefore $B' \subseteq B \backslash J_\Sigma$. Since $Y' = Y \backslash J_\Sigma$, the connected components of $\Sigma \backslash J_\Sigma \backslash Y'$ are the connected components of $\Sigma \backslash Y$ without joints. By Lemma 5.1.3, the elements of S correspond to a class of parallel links of the same sign or to half-edges incident at a vertex in $\Sigma.(B \cup Y)|Y$. Let us assume first that the elements of S correspond to a class of parallel links of the same sign. Then the elements of S correspond to links of Y of the same sign that have an end-vertex at $C(B, v)$ where v is a vertex of attachment of B in Σ i.e., $S \subseteq Y(B, v)$. It follows that there is v' vertex of attachment of B' in $\Sigma \backslash J_\Sigma$ such that $C(B, v) \backslash J_\Sigma \subseteq C(B', v')$ which implies that $Y(B, v) \backslash J_\Sigma \subseteq Y(B', v')$. Thus there is $S' \subseteq Y(B', v')$ such that S' is a bond in $\Sigma \backslash J_\Sigma.(B' \cup Y')|Y'$ for which $S \backslash J_\Sigma \subseteq S'$. Let us assume now that the elements of S correspond to half-edges incident at a vertex in $\Sigma.(B \cup Y)|Y$. Then the elements of S correspond to joints of the balancing part of Y or to links of the unbalancing part of Y that have an end-vertex at an unbalanced component of $\Sigma \backslash J_\Sigma$. Since S and S' are subsets of Y and Y' that are minimal intersections of bonds in $\Sigma.(B \cup Y)|Y$ and $\Sigma \backslash J_\Sigma.(B' \cup Y')|Y'$ respectively, we have that $S \backslash J_\Sigma \subseteq S'$.
Case 2: $Y' \subset Y \backslash J_\Sigma$
By assumption there is a joint unbalanced component in $\Sigma \backslash Y$ denoted by Σ_J^- and let B be its joint unbalanced separator. Then the bridge B' such that $B \backslash J_\Sigma \subseteq B'$ corresponds to a balanced or an unbalanced separator of $\Sigma \backslash J_\Sigma \backslash Y'$. There is only one $S \in \pi(M(\Sigma), B, Y)$ such that $S \backslash J_\Sigma \subseteq Y'$ whose elements correspond to half-edges incident at a vertex in $\Sigma.(B \cup Y)|Y$. Thereby the elements of S correspond either to links of Y that have an end-vertex at some unbalanced component of $\Sigma \backslash Y$ or to edges of the balancing part of Y in Σ. Thus there is v' vertex of attachment of B' in $\Sigma \backslash J_\Sigma \backslash Y'$ such that $B \backslash J_\Sigma \subseteq F(B', v')$. Then there is $S' \subseteq Y(B', v')$ which is a bond in $\Sigma \backslash J_\Sigma.(B' \cup Y')|Y'$ and therefore $S' \subseteq S \backslash J_\Sigma$. The same holds when $B' \subset B \backslash J_\Sigma$. □

Given a cocircuit Y of a signed-graphic matroid $M(\Sigma)$, the property of bridge-separability of a cocircuit $Y \backslash J_\Sigma$ of the minor $M(\Sigma \backslash J_\Sigma)$ is passed to Y.

Theorem 31. *If Y is a non-graphic cocircuit of an internally 4-connected quaternary signed-graphic matroid $M(\Sigma)$ and $Y' \subseteq Y \backslash J_\Sigma$ is a bridge-separable cocircuit of $M(\Sigma \backslash J_\Sigma)$, then Y is bridge-separable in $M(\Sigma)$.*

Proof. We shall show that when two bridges B'_1 and B'_2 of $M(\Sigma \backslash J_\Sigma) \backslash Y'$ are avoiding, then there are avoiding bridges B_1 and B_2 of $M(\Sigma) \backslash Y$ such that $B_i \cap B'_i \neq \emptyset, (i = 1, 2)$. Let J_Σ^+ denote the set of joints at vertices of the balanced component of $\Sigma \backslash Y$.

We distinguish the following two cases:
Case 1: B_1', B_2' are separators of the same component Σ' in $(\Sigma \backslash J_\Sigma)\backslash Y'$.
Let us assume that Σ' is unbalanced since the case that it is balanced follows similarly. Moreover, let us assume first that $Y' = Y \backslash J_\Sigma$ which implies that B_i is not a joint unbalanced separator of $\Sigma \backslash Y$. Then either B_i' is a balanced separator of the unbalanced component Σ_J' of the signed graph $\Sigma \backslash Y$ where $\Sigma' = \Sigma_J' \backslash J_\Sigma$ or B_i' is contained in the unique unbalanced separator of Σ_J'. Thus there is B_i separator of $\Sigma \backslash Y$ such that $B_i' \subseteq B_i \backslash J_\Sigma$. Due to the fact that B_1', B_2' are avoiding bridges of Y' in $M(\Sigma \backslash J_\Sigma)$, there are $S_1' \in \pi(M(\Sigma \backslash J_\Sigma), B_1', Y')$ and $S_2' \in \pi(M(\Sigma \backslash J_\Sigma), B_2', Y')$ such that $S_1' \cup S_2' = Y'$. By Lemma 5.1.3, we distinguish two cases for the elements of S_i'. If the elements of S_i' correspond to half-edges at a vertex in $\Sigma \backslash J_\Sigma.(B_i' \cup Y')|Y'$, then the elements of S_i' correspond either to links of the unbalancing part of Y' that are incident to unbalanced components of $(\Sigma \backslash J_\Sigma)\backslash Y'$ or to links of the balancing part of Y' in $\Sigma \backslash J_\Sigma$. Thereby the elements of $S_i' \cup J_\Sigma^+$ correspond to half-edges at a vertex in $\Sigma.(B_i \cup Y)|Y$ and by Lemma 5.1.3, there is $S_i \in \pi(M(\Sigma), B_i, Y)$ such that $S_i' \cup J_\Sigma^+ \subseteq S_i$. Since $Y' \cup J_\Sigma^+ = Y$, we have that $S_1 \cup S_2 = Y$. Otherwise the elements of S_i' correspond to links of the same sign incident at a vertex of attachment of B_i' in $\Sigma \backslash J_\Sigma.(B_i' \cup Y')|Y'$, which implies that there is vertex of attachment v of B_i so that the elements of S_i' correspond to links of Y that have an end-vertex at $C(B_i, v)$ in Σ. Due to avoidance of B_1' and B_2', if the elements of S_1' correspond to parallel links of the same sign at a vertex in $\Sigma \backslash J_\Sigma.(B_1' \cup Y')|Y'$ then the elements of S_2' correspond to half-edges at a vertex in $\Sigma \backslash J_\Sigma.(B_2' \cup Y')|Y'$, which implies that the elements of $S_2' \cup J_\Sigma^+$ correspond to half-edges at a vertex in $\Sigma.(B_2 \cup Y)|Y$. It follows that $S_1' = S_1$ and $S_2' \cup J_\Sigma^+ \subseteq S_2$ and therefore B_1 and B_2 are avoiding.

Let us assume now that $Y' \subset Y \backslash J_\Sigma$ and that B_1 is a joint unbalanced separator of $\Sigma \backslash Y$, therefore $B_1 \backslash J_\Sigma \subseteq B_1'$. Furthermore we assume that B_1' is an unbalanced separator and B_2' is a balanced separator of Σ' in $(\Sigma \backslash J_\Sigma)\backslash Y'$, since by Lemma 5.1.3 all other cases follow similarly. Due to the fact that B_1', B_2' are avoiding bridges of Y' in $M(\Sigma \backslash J_\Sigma)$, there are $S_1' \in \pi(M(\Sigma \backslash J_\Sigma), B_1', Y')$ and $S_2' \in \pi(M(\Sigma \backslash J_\Sigma), B_2', Y')$ such that $S_1' \cup S_2' = Y'$. By Lemma 5.1.3, the elements of S_i' correspond either to a class of parallel links of the same sign or to half-edges incident at a vertex in $\Sigma \backslash J_\Sigma.(B_i' \cup Y')|Y'$. If the elements of S_1' correspond to links in $\Sigma \backslash J_\Sigma.(B_1' \cup Y')|Y'$, then by avoidance of B_1', B_2', the elements of S_2' correspond to half-edges at a vertex in $\Sigma \backslash J_\Sigma.(B_2' \cup Y')|Y'$. Otherwise the elements of S_i' correspond to half-edges incident at a vertex in $\Sigma \backslash J_\Sigma.(B_i' \cup Y')|Y'$. In both cases, by Lemma 5.1.5, there is $S_i \in \pi(M(\Sigma), B_i, Y)$ such that $S_i' \subseteq S_i \backslash J_\Sigma$. The elements of $Y \backslash (Y' \cup J_\Sigma^+)$ are incident at an unbalanced component of $\Sigma \backslash Y$, therefore, they become joints at a vertex in $\Sigma.(B_i \cup Y)|Y$. Then they are contained in S_1 or S_2 and B_1 and B_2 are avoiding.

Case 2: B_1', B_2' are separators of different components in $(\Sigma\backslash J_\Sigma)\backslash Y'$.
Let us assume that B_1', B_2' are separators of different unbalanced components in $(\Sigma\backslash J_\Sigma)\backslash Y'$. By avoidance of B_1', B_2', there are $S_1' \in \pi(M(\Sigma\backslash J_\Sigma), B_1', Y')$ and $S_2' \in \pi(M(\Sigma\backslash J_\Sigma), B_2', Y')$ such that $S_1' \cup S_2' = Y'$. Moreover, let H_i denote the set of links of Y' in $\Sigma\backslash J_\Sigma$ that are incident to an unbalanced component of $\Sigma\backslash Y$ not containing B_i and the links of the balancing part of Y'. Then the edges of H_i are joints at a vertex in $\Sigma\backslash J_\Sigma.(B_i' \cup Y')|Y'$ while the edges of H_i and the edges of J_Σ^+ are joints at a vertex in $\Sigma.(B_i \cup Y)|Y$. By Lemma 5.1.3, there is $S_i \in \pi(M(\Sigma), B_i, Y)$ such that $H_i \cup J_\Sigma^+ \subseteq S_i$. Furthermore it holds that $H_1 \cup H_2 \cup J_\Sigma^+ = Y$, therefore we have that $S_1 \cup S_2 = Y$. The case where B_1' and B_2' are separators of an unbalanced and a balanced component in $(\Sigma\backslash J_\Sigma)\backslash Y'$ respectively, follows similarly. □

5.2 Hereditary properties through k-sums

When a signed graph Σ is k-sum, $k \in \{1, 2, 3\}$, of two signed graphs then $\Sigma\backslash J_\Sigma$ is also an l-sum, $l \leq k$, of two signed graphs. This is also the case for the corresponding signed-graphic matroids.

Lemma 5.2.1. *If $M(\Sigma)$ is a k-sum, $k \in \{1, 2, 3\}$, then $M(\Sigma\backslash J_\Sigma)$ is an l-sum, $l \leq k$.*

Proof. It is enough to show that if Σ is a k-sum, $k \in \{1, 2, 3\}$, then $\Sigma\backslash J_\Sigma$ is an l-sum, $l \leq k$, since then the result follows by Proposition 16. Suppose that $\Sigma = \Sigma_1 \oplus_k \Sigma_2$ where Σ_1, Σ_2 are signed graphs. Since $\Sigma\backslash J_\Sigma$ is obtained from Σ by removing joints, then by the definition of k-sums of signed graphs, it follows that $\Sigma\backslash J_\Sigma = \Sigma_1' \oplus_l \Sigma_2'$ where Σ_i' is a signed graph that derives from Σ_i by removing all joints. □

In the remainder of this section we will describe how the cocircuits of signed-graphic matroids, the associated bonds of signed graphs, and the properties of bridge-separability and avoidance are affected under k-sums.

5.2.1 1-sum

The inheritance of a cocircuit of a matroid $M = M_1 \oplus_1 M_2$ to either M_1 or M_2 is a direct consequence of Proposition 4.2.22 in [35]. Moreover, by the definition of bridges and Proposition 4.2.23 in [35], the following straightforward result is obtained.

Lemma 5.2.2. *If Y is a cocircuit of a matroid $M = M_1 \oplus_1 M_2$ then* *(i)* Y *is a cocircuit of exactly one, say* M_1*, and*

(ii) *the bridges of* Y *in* M *are the bridges of* Y *in* M_1 *and the elementary separators of* M_2*.*

The property of all-avoiding bridges of a cocircuit Y in $M = M_1 \oplus_1 M_2$ is inherited to a component of the 1-sum as stated in the following result.

Lemma 5.2.3. *If* Y *is a cocircuit with all-avoiding bridges of a matroid* $M = M_1 \oplus_1 M_2$ *then* Y *is a cocircuit with all-avoiding bridges in either* M_1 *or* M_2*.*

Proof. By Lemma 5.2.2, we can assume that Y is a cocircuit of M_1. Moreover, the bridges of Y in M_1 are the bridges of Y in M except for the elementary separators of M_2. Next it is proved that $\pi(M, B, Y) = \pi(M_1, B, Y)$ for every bridge B of Y in M other than the elementary separators of M_2, by showing that $\mathcal{C}^*(M.(B \cup Y)|Y) = \mathcal{C}^*(M_1.(B \cup Y)|Y)$. We have that

$$\begin{aligned}
M.(B \cup Y)|Y &= M/(E(M_1) \cup E(M_2)) - (B \cup Y)|Y \\
&= M/E(M_2)/E(M_1) - (B \cup Y)|Y \\
&= M\backslash E(M_2)/E(M_1) - (B \cup Y)|Y \\
&= M_1.(B \cup Y)|Y.
\end{aligned}$$

□

Restricting ourselves to the class of signed-graphic matroids, it is shown that there are specific relationships between special types of cocircuits in a signed-graphic matroid $M(\Sigma) = M(\Sigma_1) \oplus_1 M(\Sigma_2)$ with those of $M(\Sigma_1)$ and $M(\Sigma_2)$.

Lemma 5.2.4. *Let* Y *be a cocircuit with all-avoiding bridges of a signed-graphic matroid* $M(\Sigma) = M(\Sigma_1) \oplus_1 M(\Sigma_2)$ *where* $M(\Sigma_i)$ $(i = 1, 2)$ *signed-graphic matroid.*

(i) *If* Y *is the star of a vertex in* Σ_1 *or* Σ_2*, then it is the star of a vertex in a signed-graphic representation of* $M(\Sigma)$*.*

(ii) *If* Y *is a balancing bond in* Σ_1 *or* Σ_2 *then it is a balancing bond in* Σ*.*

Proof. For (i) suppose that Y is the star of a vertex w in Σ_1. Assume that Σ_1 contains more than one vertex. Let $\Sigma_1 \oplus_1 \Sigma_2$ be obtained by identifying a vertex $v_1 \neq w$ of Σ_1 with a vertex v_2 of Σ_2. Then Y is the star at w in $\Sigma_1 \oplus_1 \Sigma_2$ and since $M(\Sigma_1 \oplus_1 \Sigma_2) = M(\Sigma_1) \oplus_1 M(\Sigma_2) = M(\Sigma)$ the result follows. If $V(\Sigma_1) = \{w\}$, then Y is set of joints in w and Σ_1 is joint unbalanced. Thus $M(\Sigma_1) \cong M(\Sigma_1')$ where Σ_1' has exactly two vertices and Y is a star of each.

For (ii) suppose that Y is a balancing bond in Σ_1. Since Σ_2 is a balanced separator of Σ the results follows. □

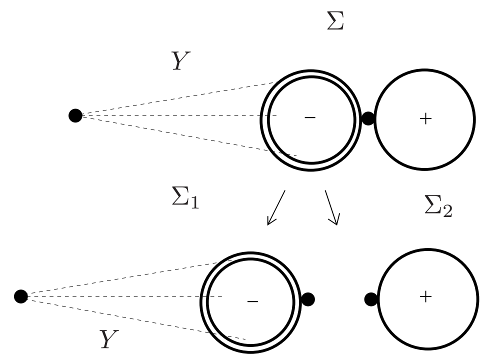


Figure 5.3: Y is the star of a vertex in $\Sigma = \Sigma_1 \oplus_1 \Sigma_2$

Theorem 32. *Let Y be a non-graphic cocircuit of a signed-graphic matroid $M(\Sigma)$ where $\Sigma = \Sigma_1 \oplus_1 \Sigma_2$. If Y is a bridge-separable cocircuit in either $M(\Sigma_1)$ or $M(\Sigma_2)$ then Y is bridge-separable in $M(\Sigma)$.*

Proof. Suppose that Y is a bridge-separable cocircuit in $M(\Sigma_1)$. By Lemma 5.2.2, the set of bridges of Y in $M(\Sigma)$ is the union of the set of bridges of Y in $M(\Sigma_1)$ and the set of elementary separators of $M(\Sigma_2)$. By Lemma 5.2.3 and since Y is bridge-separable cocircuit in $M(\Sigma_1)$, the bridges of Y in $M(\Sigma)$ which coincide with the bridges of Y in $M(\Sigma_1)$ can be partitioned into two classes so that any two bridges in the same class are avoiding. It suffices to show that the remaining bridges of Y in $M(\Sigma)$, i.e. those coinciding with the elementary separators of $M(\Sigma_2)$, are avoiding with all bridges in one of the two aforementioned classes. To prove this, let us first denote by B' an elementary separator of $M(\Sigma_2)$ and by Σ' the corresponding separator.

Let us consider first the case where Σ_1 is an unbalanced signed graph which, by the definition of 1-sum of signed graphs, implies that Σ_2 is balanced. Moreover, since $Y \subseteq E(\Sigma_1)$ and non-graphic, Σ_2 would be a subgraph of either the balanced component or an unbalanced component of $\Sigma \backslash Y$. We shall consider only the first case since the second case follows similarly. Note that in $\Sigma.(\Sigma' \cup Y)|Y$ all edges of Y are half-edges at the common vertex of Σ_1 and Σ_2. This implies that $\Sigma.(\Sigma' \cup Y)|Y$ has only one bond and, therefore, $\pi(M(\Sigma), B', Y) = \{Y\}$. Evidently, B avoids any bridge of Y in $M(\Sigma_1)$ and, thus, Y is bridge-separable in $M(\Sigma)$.

In the remaining case we have that Σ_1 is a balanced signed graph which implies that Σ_2 is an unbalanced signed graph Since $Y \subseteq E(\Sigma_1)$ and Y is a non-balancing bond of Σ, we have that Σ_2 would be contained in the unique unbalanced component of $\Sigma \backslash Y$. Then the signed graph $\Sigma.(\Sigma' \cup Y)|Y$ consists of a set of parallel positive links of Y that constitutes a bond. Therefore, $\Sigma.(\Sigma_2 \cup Y)|Y$ has only one bond consisting of the edges of Y and thus, $\pi(M(\Sigma), B', Y) = \{Y\}$ which as in the previous case implies that Y is bridge-separable in $M(\Sigma)$. $\square$

5.2.2 2-sum

Throughout this subsection we assume that we have a connected matroid $M = M_1 \oplus_2 M_2$ where M_1 and M_2 are connected matroids with ground sets $E(M_1) = X_1 \cup z$ and $E(M_2) = X_2 \cup z$. Moreover, the ground set of M is $E(M) = (E(M_1) \cup E(M_2)) - z$ so M_i is an one element extension of $M|X_i$, $(i = 1, 2)$.

Given a cocircuit Y of M, it is either contained in one of M_1, M_2 or its elements may be partitioned between them. As shown in the following result, in the former case, Y is also a cocircuit of the corresponding matroid (i.e. M_1 or M_2) while, in the latter case, the elements of Y in M_1 (M_2) and z constitute a cocircuit of M_1 (M_2). Moreover if the elements of Y are partitioned between M_1 and M_2, then the bridges of Y in M are also partitioned accordingly. Otherwise, the bridges of Y in M apart from one are inherited to either M_1 or M_2.

Lemma 5.2.5. *Let $M = M_1 \oplus_2 M_2$ be a matroid such that each M_i $(i = 1, 2)$ is connected and $E(M_i) = X_i \cup z$. If (Y_1, Y_2) is a partition of $Y \in \mathcal{C}^*(M)$ where $Y_i \subseteq E(M_i)$ then one of the following holds:*

(i) Y is a cocircuit in some M_i and its bridges are the bridges of Y in M that are contained in $E(M_i)$ and a bridge B such that $B \oplus_2 M_j$, where $j \neq i$, is a bridge of Y in M,

(ii) $Y_i \cup z$ is a cocircuit in M_i and its bridges are the bridges of Y in M that are contained in $E(M_i)$.

Proof. (i) Assume that $Y \subseteq E(M_1)$, so $Y \subseteq X_1$. Thus, $Y \in \mathcal{C}(M^*|X_1)$. Since M_1 is obtained by extending $M|X_1$ by z, $\mathcal{C}(M^*|X_1) \subseteq \mathcal{C}(M_1^*)$. Therefore, $Y \in \mathcal{C}^*(M_1)$. Moreover, it is known that $M_1 \oplus_2 M_2 = P(M_1, M_2)\backslash z$ where $P(M_1, M_2)$ denotes the matroid which is obtained by the parallel connection of M_1 and M_2. By Proposition 7.1.15 in [35], $e \in E(M_1) - z$, $(M_1 \oplus_2 M_2)\backslash e = P(M_1, M_2)\backslash p\backslash e = P(M_1, M_2)\backslash e\backslash p = P(M_1\backslash e, M_2)\backslash p = (M_1\backslash e) \oplus_2 M_2$. It follows that $(M_1 \oplus_2 M_2)\backslash Y = (M_1\backslash Y) \oplus_2 M_2$. Let us denote by B the unique bridge of Y in M_1 containing z. Since $M_1 \oplus_2 M_2$ is connected if and only if M_1 and M_2 are connected matroids, then $B \oplus_2 M_2$ is a bridge of Y in M because it is a minimal connected subset of $E(M) - Y$.

(ii) Assume that both Y_i are non-empty. By the definition of the matroid 2-sum operation we have

$$\mathcal{C}^*(M_1) = \mathcal{C}^*(M.X_1) \cup \{(C \cap X_1) \cup z : C \text{ cocircuit of } M \text{ meeting both } X_1, X_2\}. \tag{5.1}$$

Therefore, since Y is a cocircuit of M meeting both X_1 and X_2, we have that $(Y \cap X_1) \cup z \in \mathcal{C}^*(M_1)$ which is equivalent to $Y_1 \cup z \in \mathcal{C}^*(M_1)$. With respect to the bridges of Y, we shall first show that $X_i - Y_i$ is a separator of $M \backslash Y$. It is well-known that $r(X_1 - Y_1) + r(X_2 - Y_2) \geq r(X - Y)$ when $X - Y = (X_1 - Y_1) \cup (X_2 - Y_2)$ where $X_1 - Y_1, X_2 - Y_2$ disjoint sets. Therefore, if $r(X_1 - Y_1) + r(X_2 - Y_2) = r(X - Y)$ the result follows. So assume that $r(X_1 - Y_1) + r(X_2 - Y_2) > r(X - Y)$, or equivalently

$$r(X_1 - Y_1) + r(X_2 - Y_2) \geq r(X - Y) + 1 \tag{5.2}$$

Since $Y_1 \cup z \in \mathcal{C}^*(M_1)$, $X_1 - Y_1$ is a hyperplane of M_1 and $r(X_1 - Y_1) = r(M_1) - 1$. Equivalently, $r(X_1 - Y_1) = r(X_1 \cup z) - 1$. Since M_1 is connected and z is neither a loop nor a coloop of M_1, $z \in cl(X_1)$ and $r(X_1 \cup z) = r(X_1)$. Therefore, $r(X_1 - Y_1) = r(X_1) - 1$. Similarly, $r(X_2 - Y_2) = r(X_2) - 1$. Then equation (5.2) becomes $r(X_1) - 1 + r(X_2) - 1 \geq r(X - Y) + 1$. Furthermore, $Y \in \mathcal{C}^*(M)$, so $X - Y$ is a hyperplane of M and $r(X - Y) = r(M) - 1$. Thus, $r(X_1) + r(X_2) \geq r(M) + 2$ which is a contradiction since (X_1, X_2) is an exact 2-separation with $r(X_1) + r(X_2) = r(M) + 1$. By definition of matroidal 2-sum, $M_1 \backslash (Y_1 \cup z) = M|(X_1 - Y_1)$. Thereby, $M \backslash Y = M|(X_1 - Y_1) \oplus_1 M|(X_2 - Y_2) = M_1 \backslash (Y_1 \cup z) \oplus_1 M_2 \backslash (Y_2 \cup z)$. Therefore, every bridge of Y in M is contained either to $E(M_1)$ or $E(M_2)$. $\square$

The following four technical lemmas are needed for the proofs of Theorem 33 and Theorem 34.

Lemma 5.2.6. *Let $M = M_1 \oplus_2 M_2$ be a matroid, $Y \in \mathcal{C}^*(M)$ and each M_i $(i = 1, 2)$ is connected with $E(M_i) = X_i \cup z$. If $Y \in \mathcal{C}^*(M_i)$ and B a bridge of Y in M_i then either:*

(i) $z \notin B$ and $\pi(M_i, B, Y) = \pi(M, B, Y)$, or

(ii) $z \in B$ and $\pi(M_i, B, Y) = \pi(M, B \oplus_2 M_j, Y)$, $(i \neq j)$.

Proof. For the proof that follows assume that $i = 1$.
(i) By Lemma 5.2.5 (i) B is a bridge of Y in M. Enough to show that $Y_1 \in \mathcal{C}^*(M.(B \cup Y)|Y)$ if and only if $Y_1 \in \mathcal{C}^*(M_1.(B \cup Y)|Y)$. Assume that $Y_1 \in \mathcal{C}^*(M.(B \cup Y)|Y)$, then there exists $Y_2 \in \mathcal{C}^*(M.(B \cup Y))$ such that $C_1^* = Y_2 \cap Y$. It follows that $Y_2 \in \mathcal{C}^*(M)$. Moreover, $Y_2 \subseteq X_1$, so $Y_2 \in \mathcal{C}^*(M.X_1)$. By (5.1) we have that $\mathcal{C}^*(M.X_1) = \mathcal{C}^*(M_1.X_1)$, thus, $Y_1 \in \mathcal{C}^*(M_1.X_1)$ which implies $Y_1 \in \mathcal{C}^*(M_1)$.

Since $Y_2 \subseteq B \cup Y$ then $Y_2 \in \mathcal{C}^*(M_1.(B \cup Y))$. Furthermore $C_1^* = Y_2 \cap Y$ which implies that either $Y_1 \in \mathcal{C}^*(M_1.(B \cup Y)|Y)$ or Y_1 contains a member of $\mathcal{C}^*(M_1.(B \cup Y)|Y)$. However, in the latter case, there exists $Y_3 \in \mathcal{C}^*(M_1.(B \cup Y))$ such that $Y_3 \cap Y \subset Y_2 \cap Y$ implying that $Y_3 \in \mathcal{C}^*(M.(B \cup Y))$ which is a contradiction to

our initial assumption that $Y_2 \cap Y = Y_1 \in \mathcal{C}^*(M.(B \cup Y)|Y)$. Reversing the above arguments it is proved that if $Y_1 \in \mathcal{C}^*(M_1.(B \cup Y)|Y)$ then $Y_1 \in \mathcal{C}^*(M.(B \cup Y)|Y)$.
(ii) We will show that $\mathcal{C}^*(M.(B \oplus_2 M_2 \cup Y)|Y) = \mathcal{C}^*(M_1.(B \cup Y)|Y)$. Suppose that $Y_1 \in \mathcal{C}^*(M.(B \oplus_2 M_2 \cup Y)|Y)$, then there exists $Y_2 \in \mathcal{C}^*(M.((B \oplus_2 M_2) \cup Y))$ such that $Y_1 = Y_2 \cap Y$. Therefore, $Y_2 \in \mathcal{C}^*(M)$. Since $z \notin g'$, we distinguish two cases as regards to g'. Either $Y_2 \subseteq X_1$ or g' meets both X_1 and X_2. If $Y_2 \subseteq X_1$, since $Y_2 \in \mathcal{C}^*(M)$, $Y_2 \in \mathcal{C}^*(M.X_1) = \mathcal{C}^*(M_1.X_1)$; moreover, $Y_2 \subseteq B \cup Y$, therefore, $Y_2 \in \mathcal{C}^*(M_1.(B \cup Y))$. Hence, it follows that $Y_2 \cap Y = Y_1 \in \mathcal{C}^*(M_1.(B \cup Y)|Y)$. Consider now the case that g' meets both X_1, X_2. Since $Y_2 \in \mathcal{C}^*(M)$, by (5.1) we have $(Y_2 \cap X_1) \cup z \in \mathcal{C}^*(M_1)$; moreover, $z \in B$ so $Y_2 \cap X_1 = Y_2 \cap (B \cup Y)$. Thereby, $[Y_2 \cap (B \cup Y)] \cup z \in \mathcal{C}^*(M_1)$ and $[Y_2 \cap (B \cup Y)] \cup z \subseteq B \cup Y$. It follows that $[Y_2 \cap (B \cup Y)] \cup z \in \mathcal{C}^*(M_1.(B \cup Y))$. Since $Y_1 = Y_2 \cap Y$, we have that $Y_1 \in \mathcal{C}^*(M_1.(B \cup Y)|Y)$.

□

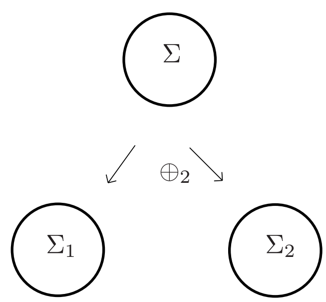


Figure 5.4: $M(\Sigma) = M(\Sigma_1) \oplus_2 M(\Sigma_2)$

Lemma 5.2.7. *Let $M = M_1 \oplus_2 M_2$ be a matroid such that each M_i $(i = 1, 2)$ is connected with $E(M_i) = X_i \cup z$ and (Y_1, Y_2) a partition of $Y \in \mathcal{C}^*(M)$. If $Y_i \subseteq E(M_i)$ such that $Y_i \cup z \in \mathcal{C}^*(M_i)$ and B is a bridge of Y in M where $B \subseteq E(M_1)$ then there exists $S \in \pi(M, B, Y)$ such that $Y_2 \subseteq S$.*

Proof. By the relationship of cocircuits of a M and its contraction minor $M' = M.(B \cup Y)$, it is evident that Y is a cocircuit of M. If for any cocircuit Y' of M' we have that either $Y_2 \subseteq Y'$ or $Y_2 \cap Y' = \emptyset$, then the lemma holds by the definition regarding the elements of $\pi(M, B, Y)$. Thus, it remains to show there exists no Y' such that $Y_2 \not\subseteq Y'$ or $Y_2 \cap Y' \neq \emptyset$. If such an Y' existed then it would be a cocircuit of M as well; thus, by (5.1) the set $(Y \cap X_2) \cup z = Y \cup z$ should be a cocircuit of M_2 which contradicts the minimality of Y_2. □

Example 5.2.1. *Consider the signed graph Σ which is depicted in Figure 5.5. The signed graph Σ is 1-vertex 2-sum of Σ_1 and Σ_2 (Figure 5.4) where the signed graphs Σ_1 and Σ_2 are depicted in Figures 5.6(a) and 5.6(b) respectively. The signed-graphic matroid $M(\Sigma) = M(\Sigma_1) \oplus_2 M(\Sigma_2)$ has a non-graphic cocircuit $Y = \{4, 5, -5, -3\}$ which is bridge-separable and corresponds to a double bond in Σ. The bridges of Y in $M(\Sigma)$ are $B_1 = \{1, 2, 3, -1, -2, -6\}$ and $B_2 = \{-4\}$. Then $\pi(M(\Sigma), B_1, Y) = \{\{-3, -5\}, \{4\}, \{5\}\}$ and $\pi(M(\Sigma), B_2, Y) = \{\{-3, 4\}, \{-5, 5\}\}$.*

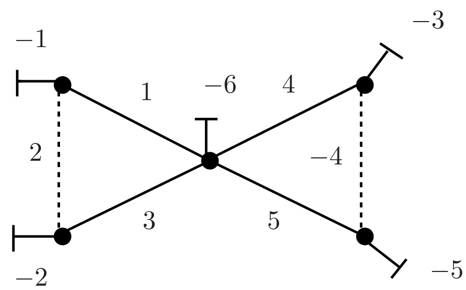
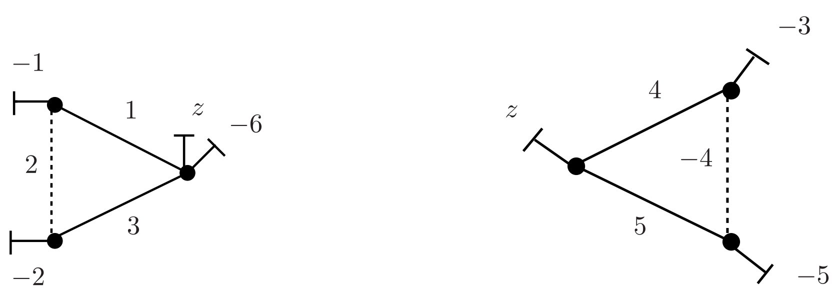


Figure 5.5: The signed graph Σ

The double bond Y of Σ, which is contained in $E(\Sigma_2)$, is a double bond of Σ_2. Moreover, the cocircuit Y of $M(\Sigma_2)$ is graphic and bridge-separable. The bridges of Y in $M(\Sigma_2)$ are $B'_1 = \{z\}$ and $B_2 = \{-4\}$. Then $\pi(M(\Sigma_2), B'_1, Y) = \{\{-3, -5\}, \{4\}, \{5\}\}$ and $\pi(M(\Sigma_2), B_2, Y) = \{\{-3, 4\}, \{-5, 5\}\}$. Note that $B_1 = B_1 \oplus_2 B'_1$.



(a) The signed graph Σ_1 (b) The signed graph Σ_2

Figure 5.6: The cocircuit $Y \subseteq E(\Sigma_2)$

Example 5.2.2. *Consider the signed graph Σ which is depicted in Figure 5.7. The signed graph Σ is 1-vertex 2-sum of Σ_1 and Σ_2 (Figure 5.4) where the signed graphs Σ_1 and Σ_2 are depicted in Figures 5.8(a) and 5.8(b) respectively. The signed-graphic matroid $M(\Sigma) = M(\Sigma_1) \oplus_2 M(\Sigma_2)$ has a non-graphic cocircuit $Y =$*

$\{2, -2, 4, 5, -6, -1\}$ *which is bridge-separable and corresponds to a double bond in* Σ. *The bridges of* Y *in* $M(\Sigma)$ *are* $B_1 = \{1\}$, $B_2 = \{3\}$ *and* $B_3 = \{-3, -4, -5, -7\}$. *Then*

$$\pi(M(\Sigma), B_1, Y) = \{\{-1\}, \{2\}, \{-2, 4, -6, 5\}\}$$
$$\pi(M(\Sigma), B_2, Y) = \{\{2\}, \{-2\}, \{-1, 4, -6, 5\}\}$$
$$\pi(M(\Sigma), B_3, Y) = \{\{-1-2, 2, -6\}, \{4\}, \{5\}\}.$$

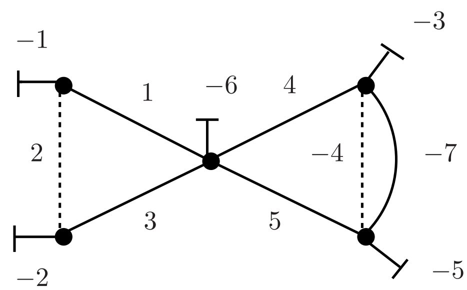


Figure 5.7: The signed graph Σ

Let (Y_1, Y_2) *be a partition of* Y *such that* $Y_1 = \{1, 2, -2\}$ *and* $Y_2 = \{4, -6, 5\}$. *Then* $Y_1 \cup z = \{-1, -2, 2, z\}$ *is a bridge-separable cocircuit of* $M(\Sigma_1)$ *where* $B_1 = \{1\}$ *and* $B_2 = \{3\}$ *are the bridges of* $Y_1 \cup z$ *in* $M(\Sigma_1)$. *Moreover,* $Y_2 \cup z = \{4, 5, -6, z\}$ *is a bridge-separable cocircuit of* $M(\Sigma_2)$ *where* B_3 *is the unique bridge of* Y *in* $M(\Sigma_2)$. *Then*

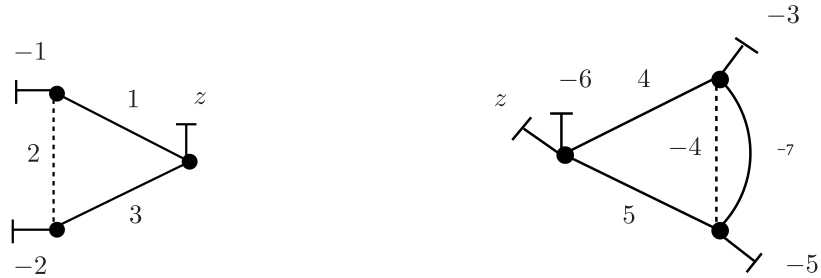
$$\pi(M(\Sigma_1), B_1, Y_1 \cup z) = \{\{-1\}, \{2\}, \{-2, z\}\}$$
$$\pi(M(\Sigma_1), B_2, Y_1 \cup z) = \{\{-2\}, \{2\}, \{-1, z\}\}$$
$$\pi(M(\Sigma_1), B_3, Y_1 \cup z) = \{\{-6, z\}, \{4\}, \{5\}\}.$$

Note that if B is a bridge of Y in M such that $B \subseteq E(M_i)$ and $Y_i \cup z \in \mathcal{C}^*(M_i)$ for some $i \in \{1, 2\}$, then by Lemma 5.2.5 we have that B is a bridge of $Y_i \cup z$ in M_i.

Lemma 5.2.8. *Let* $M = M_1 \oplus_2 M_2$ *be a matroid such that each* M_i $(i = 1, 2)$ *is connected with* $E(M_i) = X_i \cup z$ *and* (Y_1, Y_2) *a partition of* $Y \in \mathcal{C}^*(M)$. *If* $Y_i \subseteq E(M_i)$ *such that* $Y_i \cup z \in \mathcal{C}^*(M_i)$, B *is a bridge of* Y *in* M *where* $B \subseteq E(M_1)$ *and* $S \in \pi(M, B, Y)$, *then*

(i) $S \in \pi(M_1, B, Y_1 \cup z)$, *if* $Y_2 \nsubseteq S$, *or*

(ii) $(S - Y_2) \cup z \in \pi(M_1, B, Y_1 \cup z)$, *otherwise.*



(a) $Y_1 \cup z$ cocircuit of $M(\Sigma_1)$ (b) $Y_2 \cup z$ cocircuit of $M(\Sigma_2)$

Figure 5.8: The signed graphs Σ_1 (a) and Σ_2 (b)

Proof. By Lemma 5.2.7 and the definition of $\pi(M, B, Y)$, for any $C \in \mathcal{C}^*(M.(B \cup Y)|Y)$ we can distinguish two cases:

<u>Case 1:</u> $C \cap Y_2 = \emptyset$.

We have that $C \subseteq Y_1$ and there exists $C' \in \mathcal{C}^*(M.(B \cup Y))$ such that $C = C' \cap Y$. Combining that with the assumption that $C \cap Y_2 = \emptyset$ we have that $C' \cap Y_2 = \emptyset$, which in turn implies that $C' \subseteq B \cup Y_1 \subseteq X_1$. Therefore, by the fact that $\mathcal{C}^*(M.X_1) = \mathcal{C}^*(M_1.X_1)$, we have that $C' \in \mathcal{C}^*(M_1.(B \cup Y_1))$ implying that $C' \in \mathcal{C}^*(M_1.(B \cup Y_1 \cup z))$. Moreover, since $C \subseteq Y_1$ and $z \notin C'$ we have that $C = C' \cap Y = C' \cap (Y_1 \cup z)$ and, thereby, $C \in \mathcal{C}^*(M_1.(B \cup Y_1 \cup z)|Y_1 \cup z)$.

<u>Case 2:</u> $Y_2 \subseteq C$.

There exists $C' \in \mathcal{C}^*(M.(B \cup Y))$ such that $C = C' \cap Y$ then $C' \in \mathcal{C}^*(M)$. Since B is a bridge of $Y_1 \cup z$ in M_1, it follows that $C \cap Y_1 \neq \emptyset$. Thereby C' meets both X_1, X_2 implying that $(C' \cap X_1) \cup z \in \mathcal{C}^*(M_1)$. Since $z \notin C'$ and $C' \subseteq B \cup Y$, we have that $C' \cap X_1 = C' \cap (B \cup Y_1)$. Thus, $(C' \cap (B \cup Y_1)) \cup z \in \mathcal{C}^*(M_1)$ and moreover, $(C' \cap (B \cup Y_1)) \cup z \in \mathcal{C}^*(M_1.(B \cup Y_1 \cup z))$. It holds that $(C' \cap (B \cup Y_1)) \cup z$ $= (C - Y_2) \cup z$ so $(C - Y_2) \cup z \in \mathcal{C}^*(M_1.(B \cup Y_1 \cup z))$. Due to the fact that $C \in \mathcal{C}^*(M.(B \cup Y)|Y)$ and $C' \cap (B \cup Y_1)) \cup z$, it follows that $(C - Y_2) \cup z$ $\in \mathcal{C}^*(M.(B \cup Y_1 \cup z)|Y_1 \cup z)$. □

Following the lines of the proof of Lemma 5.2.8, the following result can be obtained.

Lemma 5.2.9. *Let $M = M_1 \oplus_2 M_2$ be a matroid such that each M_i $(i = 1, 2)$ is connected with $E(M_i) = X_i \cup z$ and (Y_1, Y_2) a partition of $Y \in \mathcal{C}^*(M)$. If $Y_i \subseteq E(M_i)$ such that $Y_i \cup z \in \mathcal{C}^*(M_i)$, B is a bridge of Y in M where $B \subseteq E(M_1)$ and $S \in \pi(M_1, B, Y_1 \cup z)$ then*

(i) $S \in \pi(M, B, Y)$ if $z \notin S$, or

(ii) $(S - z) \cup Y_2 \in \pi(M, B, Y)$, otherwise.

Example 5.2.3. *Consider the signed graph Σ which is depicted in Figure 5.9. The signed graph Σ is 2-vertex 2-sum of Σ_1 and Σ_2 along z where the signed graphs Σ_1 and Σ_2 are depicted in Figures 5.10(a) and 5.10(b) respectively. The signed-graphic matroid $M(\Sigma) = M(\Sigma_1) \oplus_2 M(\Sigma_2)$ has a non-graphic cocircuit $Y = \{7, 4, 8, 11, -4, -6, -9, -10, -11, -13, \}$ which is bridge-separable and corresponds to an unbalancing bond in Σ. The bridges of Y in $M(\Sigma)$ are $B_1 = \{5\}$, $B_2 = \{6\}$, $B_3 = \{-5\}$, $B_4 = \{-7\}$ and $B_5 = \{1, 2, 3, 9, 10, -1, -2, -3, -8, -12\}$. Then*

$$\pi(M(\Sigma), B_1, Y) = \{\{-9, -10\}, \{-4, 4, 11, -13, 7, 8, -6, -11\}\}$$
$$\pi(M(\Sigma), B_2, Y) = \{\{7, 8\}, \{-6, -11, 4, 11, -4, -9, -10, -13\}\}$$
$$\pi(M(\Sigma), B_3, Y) = \{\{-4, -13, 11\}, \{4, -9, -10, 7, 8, -6, -11\}\}$$
$$\pi(M(\Sigma), B_4, Y) = \{\{-11, 7\}, \{-6, 8, 4, 11, -4, -9, -10, -13\}\}$$
$$\pi(M(\Sigma), B_5, Y) = \{\{-4, -9\}, \{4\}, \{-10\}, \{-13, 7, 8, -6, -11\}, \{11\}\}$$

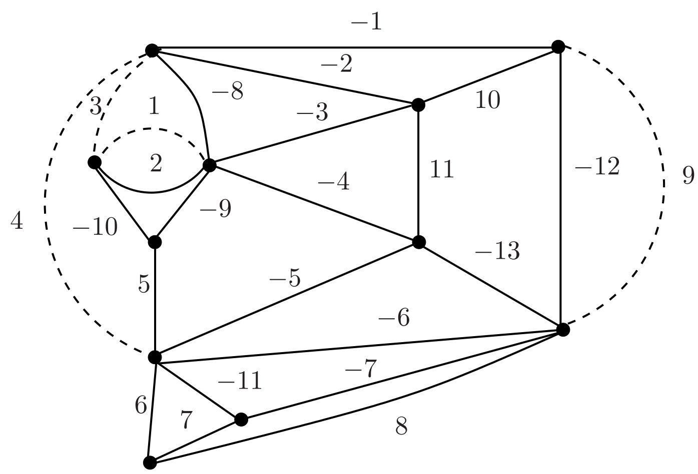
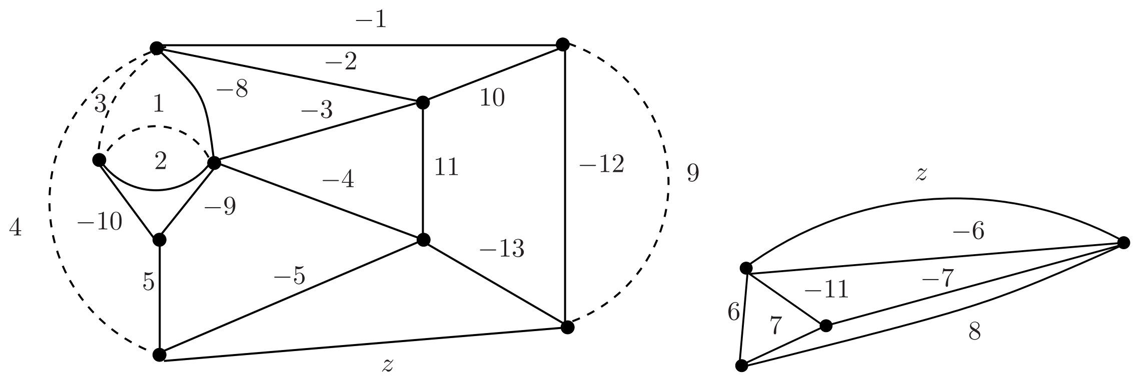


Figure 5.9: The signed graph Σ

Let (Y_1, Y_2) be a partition of Y such that $Y_1 = \{4, 11, -4, -9, -10, -13, \}$ and $Y_2 = \{7, 8, -6, -11\}$. Then $Y_1 \cup z$ is a bridge-separable cocircuit of $M(\Sigma_1)$ where $B_1 = \{1\}$, $B_3 = \{-5\}$ and $B_5 = \{1, 2, 3, 9, 10, -1, -2, -3, -8, -12\}$ are the bridges of $Y_1 \cup z$ in $M(\Sigma_1)$. Then

$$\pi(M(\Sigma_1), B_1, Y_1 \cup z) = \{\{-9, -10\}, \{-4, 4, 11, -13, z\}\}$$
$$\pi(M(\Sigma_1), B_3, Y_1 \cup z) = \{\{-4, -13, 11\}, \{4, -9, -10, z\}\}$$
$$\pi(M(\Sigma_1), B_5, Y_1 \cup z) = \{\{-4, -9\}, \{4\}, \{-10\}, \{-13, z\}, \{11\}\}.$$

Moreover, $Y_2 \cup z$ is a bridge-separable cocircuit of $M(\Sigma_2)$ where $B_2 = \{6\}$ and



(a) $Y_1 \cup z$ cocircuit of $M(\Sigma_1)$ (b) $Y_2 \cup z$ cocircuit of $M(\Sigma_2)$

Figure 5.10: The signed graphs Σ_1 (a) and Σ_2 (b)

$B_4 = \{-7\}$ *are the bridges of* Y *in* $M(\Sigma_2)$. *Then*

$$\pi(M(\Sigma_2), B_2, Y_2 \cup z) = \{\{7, 8\}, \{-6, -11, z\}\}$$
$$\pi(M(\Sigma_1), B_4, Y_2 \cup z) = \{\{-11, 7\}, \{-6, 8, z\}\}.$$

The property of all-avoiding bridges of a cocircuit in $M = M_1 \oplus_2 M_2$ is inherited in a specific way in the cocircuits of M_1 and M_2 as described in the following theorem.

Theorem 33. *Let* $M = M_1 \oplus_2 M_2$ *be a matroid such that each* M_i $(i = 1, 2)$ *is connected with* $E(M_i) = X_i \cup z$ *and* (Y_1, Y_2) *a partition of* $Y \in \mathcal{C}^*(M)$. *If* Y *has all-avoiding bridges in* M *and* $Y_i \subseteq E(M_i)$ *then one of the following holds:*

(i) Y *is a cocircuit with all-avoiding bridges in either* M_1 *or* M_2,

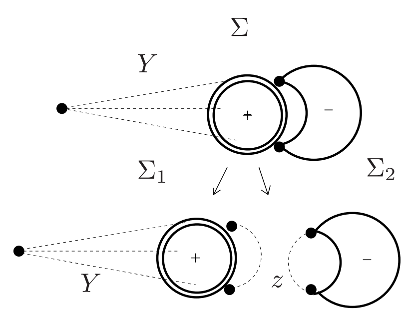
(ii) $Y_i \cup z$ *is a cocircuit with all-avoiding bridges in* M_i.

Proof. We shall show that (i) holds when either Y_1 and Y_2 is empty while (ii) holds when both Y_1 and Y_2 are non-empty.

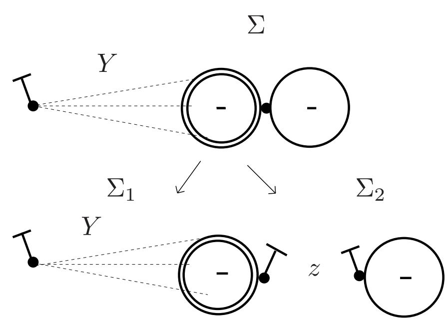
For (i) assume that $Y \subseteq X_1$. By Lemma 5.2.5(i), Y is a cocircuit of M_1. Let B_1 and B_2 be two bridges of Y in M and B be the bridge of Y in M_1 that contains z. Then, by Lemma 5.2.5(i), it also follows that $B \oplus_2 M_2$ is a bridge of Y in M. Assume first that the bridges B_1 and B_2 of Y in M are distinct from $B \oplus_2 M_2$; then, by Lemma 5.2.5(i), B_1 and B_2 are bridges of Y in M_1 distinct from B. By Lemma 5.2.6(i), it holds that $\pi(M, B_j, Y) = \pi(M_1, B_j, Y)$ $(j = 1, 2)$ and, therefore, B_1 and B_2 are avoiding bridges of Y in M_1. Assume now that one of B_1 or B_2 is $B \oplus_2 M_2$, say B_1. Then, by Lemma 5.2.5(i), B_2 is a bridge of Y in M_1 and, by Lemma 5.2.6, B is avoiding with any other bridge of Y in M_1.

For (ii), let us assume that $i = 1$, since the same arguments can be used for $i = 2$. By Lemma 5.2.5(ii), $Y_1 \cup z$ is a cocircuit of M_1. We shall show that any two bridges B_1 and B_2 of Y in M such that $B_1, B_2 \subseteq E(M_1)$ are avoiding bridges of $Y_1 \cup z$ in M_1. Since B_1, B_2 are avoiding bridges of Y in M, there are $S_1 \in \pi(M, B_1, Y)$ and $S_2 \in \pi(M, B_2, Y)$ such that $S_1 \cup S_2 = Y$. By Lemma 5.2.7, Y_2 is either contained in one of S_1, S_2 or in both of them. Let us assume that $Y_2 \subseteq S_1$. By Lemma 5.2.8, there are $S_1' \in \pi(M_1, B_1, Y_1 \cup z)$ such that $S_1' = (S_1 - Y_2) \cup z$ and $S_2' \in \pi(M_1, B_2, Y_1 \cup z)$ such that $S_2' = S_2$ and therefore, $S_1' \cup S_2' = (Y_1 \cup z)$. If $Y_2 \in S_1 \cap S_2$, then by Lemma 5.2.8, there are $S_1' \in \pi(M_1, B_1, Y_1 \cup z)$ such that $S_1' = (S_1 - Y_2) \cup z$ and $S_2' \in \pi(M_1, B_2, Y_1 \cup z)$ such that $S_2' = (S_2 - Y_2) \cup z$ and, therefore, $S_1' \cup S_2' = Y_1 \cup z$. □

The operation of 2-sum preserves star bonds and unbalancing bonds in the manner described in the next two results.



(a) Σ is 2-vertex 2-sum of Σ_1 and Σ_2



(b) Σ is 1-vertex 2-sum of Σ_1 and Σ_2

Figure 5.11: Y is the star of a vertex in Σ

Lemma 5.2.10. *Let $M(\Sigma) = M(\Sigma_1) \oplus_2 M(\Sigma_2)$ be a signed-graphic matroid such that each $M(\Sigma_i)$ $(i = 1, 2)$ is connected signed-graphic with $E(M(\Sigma_i)) = X_i \cup z$. If $Y \in \mathcal{C}^*(M(\Sigma))$ has all-avoiding bridges in $M(\Sigma)$ and $Y \subseteq E(M(\Sigma_1))$, then:*

(i) if Y is the star of a vertex in Σ_1, then it is the star of a vertex in a signed-graphic representation of $M(\Sigma)$.

(ii) if Σ is a 2-vertex 2-sum of Σ_1 and Σ_2 and Y is a balancing bond of the unbalanced Σ_1, then Y is a balancing bond in Σ.

(iii) if Σ is a 1-vertex 2-sum of Σ_1 and Σ_2, then Y cannot be a balancing bond in Σ_1.

Proof. For (i), suppose that Y is the star of a vertex w in Σ_1. Assume first that Σ is the 2-vertex 2-sum of Σ_1 and Σ_2 and v_1 and v_2 be the vertices of Σ_1 which are the end-vertices of z. Since Y is a bond of Σ, it follows that $z \notin Y$, therefore Y cannot be the star of v_1 or v_2 (see Figure 5.11(a)). Assume now that Σ is the 1-vertex 2-sum of Σ_1 and Σ_2 and moreover, assume that v_1 is the vertex of Σ_1 which is identified with a vertex of Σ_2. Since Y is a bond in Σ, it follows that $z \notin Y$, and therefore Y cannot be the star of the end-vertex v_1 of z in Σ_1 (see Figure 5.11(b)). In both cases, since $M(\Sigma_1 \oplus_2 \Sigma_2) = M(\Sigma_1) \oplus_2 M(\Sigma_2) = M(\Sigma)$, the result follows.

For (ii), since Y is a balancing bond in Σ_1, by the definition of the 2-vertex 2-sum, Σ_1 is the unbalanced signed graph and Σ_2 is the balanced one. Moreover, z can be considered as a positive link (after applying switchings at vertices if necessary) in both Σ_1 and Σ_2. Due to the minimality of Y (i.e. being a cocircuit in $M(\Sigma_1)$), there exists a series of switchings at the vertices of Σ_1 such that the edges of $\Sigma_1 \backslash Y$ become positive while the edges in Y become negative. Since switching at the vertices of a signed graph does not alter the associated matroid, by the definition of 2-vertex 2-sum of two signed graphs, it follows that we can assume that the only negative edges in Σ are the edges of Y and, therefore, Y is a balancing bond in Σ.

For (iii), by Lemma 5.2.5(i), Y is a cocircuit of $M(\Sigma_1)$. By way of contradiction assume that Y is a balancing bond in Σ_1. Applying switchings at the vertices of Σ_1 all edges of $\Sigma_1 \backslash Y$ become positive. Then by the definition of 1-vertex 2-sum of two signed graphs, Σ_2 is unbalanced and, since $M(\Sigma_2)$ is connected, $\Sigma_2 \backslash z$ should also be unbalanced. Thus, Y is not a minimal set whose deletion increases the number of balanced components in Σ, which contradicts the fact that Y is a cocircuit of $M(\Sigma)$. □

Lemma 5.2.11. *Let $M(\Sigma) = M(\Sigma_1) \oplus_2 M(\Sigma_2)$ be a signed-graphic matroid such that each $M(\Sigma_i)$ $(i = 1, 2)$ is connected signed-graphic with $E(M(\Sigma_i)) = X_i \cup z$ and (Y_1, Y_2) a partition of $Y \in \mathcal{C}^*(M(\Sigma))$ with Y_1 and Y_2 being nonempty. If Y has all-avoiding bridges in $M(\Sigma)$ and $Y_i \subseteq E(M(\Sigma_i))$ then:*

(i) if $Y_i \cup z$ is the star of a vertex in each Σ_i, then Y is the star of a vertex in a signed-graphic representation of $M(\Sigma)$.

(ii) if Σ is the 1-vertex 2-sum of Σ_1 and Σ_2 and $Y_i \cup z$ is a balancing bond in each Σ_i then Y is a balancing bond in Σ.

(iii) If Σ is the 2-vertex 2-sum of Σ_1 and Σ_2 then $Y_i \cup z$ cannot be a balancing bond in each Σ_i.

(iv) *if Σ is the 1-vertex 2-sum of Σ_1 and Σ_2 and $Y_i \cup z$ is the star of a vertex in one of Σ_1, Σ_2 and a balancing bond in the other, then Y is a balancing bond in Σ.*

(v) *if Σ is the 2-vertex 2-sum of Σ_1 and Σ_2 then $Y_i \cup z$ cannot be the star of a vertex in one of Σ_1, Σ_2 and a balancing bond in the other.*

Proof. We distinguish two cases:
Case 1: Σ is the 2-vertex 2-sum of Σ_1 and Σ_2.
Let us denote by v_1 and v_2 the common vertices of $\Sigma|X_1$ and $\Sigma|X_2$ in Σ and by v_i^1 and v_i^2 $(i = 1, 2)$ the vertices of Σ_1 and Σ_2, respectively, which are identified in order to form v_i in Σ (namely, v_1^1 and v_1^2 are identified with v_2^1 and v_2^2, respectively, in the 2-sum operation).
For (i), since $Y_i \cup z$ is the star of a vertex in Σ_i, then it is the star of v_i^1 or v_i^2. If $Y_i \cup z$ is the star of v_i^1 (resp. v_i^2) in Σ_i, then by the 2-vertex 2-sum operation, Y is the star of v_1 (resp. v_2) in Σ. If $Y_1 \cup z$ is the star of v_1^1 in Σ_1 and $Y_2 \cup z$ is the star of v_2^2 in Σ_2, then for the signed graph Σ' which is the twisted signed graph of Σ about $\{v_1, v_2\}$ we have that $M(\Sigma') = M(\Sigma)$ and Y is the star of a vertex of Σ'; similarly is treated the case in which $Y_1 \cup z$ is the star of v_1^2 in Σ_1 and $Y_2 \cup z$ is the star of v_2^1 in Σ_2 .
For (iii), let use first suppose that $Y_i \cup z$ is a balancing bond in each Σ_i. Then both Σ_1 and Σ_2 must be unbalanced which is in contradiction with the definition of the 2-vertex 2-sum of two signed graphs.
For (v) suppose that $Y_i \cup z$ is the star of a vertex in one of Σ_1, Σ_2 and a balancing bond in the other. This implies that at least one of Σ_1, Σ_2 is unbalanced. Suppose w.l.o.g. that Σ_1 is unbalanced and Σ_2 is balanced. Then $Y_1 \cup z$ is a balancing bond in Σ_1 and $Y_2 \cup z$ is the star of v_2^1 or v_2^2, say v_2^1, in Σ_2. Since Σ_2 is balanced, we may assume that the links of Y_2 which are incident to v_2^1 are positive. Thus, $Y_1 \subset Y$ is a bond of Σ which contradicts the hypothesis that Y is a bond of Σ.
Case 2: Σ is the 1-vertex 2-sum of Σ_1 and Σ_2.
Let v_1 and v_2 be the vertices of Σ_1, and Σ_2, respectively, which are identified in order to form the vertex v in Σ in the 2-sum operation.
For (i), since $Y_i \cup z$ is the star of v_i in Σ_i, by the definition of the 1-vertex 2-sum operations, it follows that Y is the star of v in Σ.
For (ii), since $Y_i \cup z$ is a balancing bond in Σ_i, we perform switchings at the vertices of Σ_i so that all edges of $\Sigma_i \backslash (Y_i \cup z)$ become positive. Then, by minimality of $Y_i \cup z$ (i.e. it is a cocircuit in Σ_i), its edges are negative in Σ_i and, therefore, Y is a balancing bond in Σ.
For (iv) it follows by the definitions of 1-vertex 2-sum and a balancing bond. $\square$

As the following result shows, for a signed-graphic matroid M such that $M = M_1 \oplus_2 M_2$ where M_i $(i = 1, 2)$ is signed-graphic, the existence of a bridge-separable cocircuit in M_1 or M_2 induces the existence of a bridge-separable cocircuit in M.

Theorem 34. *Let $M(\Sigma) = M(\Sigma_1) \oplus_2 M(\Sigma_2)$ be a signed-graphic matroid such that each $M(\Sigma_i)$ $(i = 1, 2)$ is connected and signed-graphic with $E(M(\Sigma_i)) = X_i \cup z$. Moreover, let (Y_1, Y_2) be a partition of a cocircuit Y of $M(\Sigma)$ where $Y_i \subseteq E(M(\Sigma_i))$. If Y is a bridge-separable cocircuit of some $M(\Sigma_i)$ or, for all i, $Y_i \cup z$ is a bridge-separable cocircuit of $M(\Sigma_i)$, then Y is bridge-separable in $M(\Sigma)$.*

Proof. We distinguish the following two cases:
<u>Case 1:</u> $Y \subseteq E(M(\Sigma_i))$ for some i.
Suppose that $i = 1$. Since Y is a cocircuit of $M(\Sigma)$, it holds that $z \notin Y$. Thus there is a bridge B of Y in $M(\Sigma_1)$ such that $z \in B$. By Lemma 5.2.5(i), the bridges of Y in $M(\Sigma)$ are the bridges of Y in $M(\Sigma_1)$, apart from B, and $B \oplus_2 M(\Sigma_2)$. Moreover, by Lemma 5.2.6, for each bridge B' of Y in $M(\Sigma_1)$ that does not contain z, it holds that $\pi(M(\Sigma_1), B', Y) = \pi(M(\Sigma), B', Y)$, while for B it holds that $\pi(M(\Sigma_1), B, Y) = \pi(M(\Sigma), B \oplus_2 M(\Sigma_2), Y)$. Combining the existence of a partition of the bridges of Y in $M(\Sigma_1)$ into two classes each consisting of all-avoiding bridges with the above relations, we get a partition of the bridges of Y in $M(\Sigma)$ into two classes each consisting of all-avoiding bridges, where the bridge B is replaced by $B \oplus_2 M(\Sigma_2)$.
<u>Case 2:</u> Each Y_i is nonempty.
Since there is a partition of the bridges of $Y_i \cup z$ $i = 1, 2$ in $M(\Sigma_i)$ into two classes $\mathscr{U}_i^1$ and $\mathscr{U}_i^2$ such that any two bridges in the same class are avoiding, it is shown first that any two bridges in the same class $\mathscr{U}_i^j$ $(i, j = 1, 2)$ are also avoiding bridges of Y in $M(\Sigma)$. Let B_1 and B_2 be two arbitrary bridges in $\mathscr{U}_1^1$, then $B_1, B_2 \subseteq E(M(\Sigma_1))$. It follows that there are $S_1 \in \pi(M(\Sigma_1), B_1, Y_1 \cup z)$ and $S_2 \in \pi(M(\Sigma_1), B_2, Y_1 \cup z)$ such that $S_1 \cup S_2 = Y_1 \cup z$. Then either z belongs to at least one of S_1, S_2 or $z \in S_1 \cap S_2$. In the first case, let us assume that $z \in S_1$ and $z \notin S_2$. By Lemma 5.2.5(ii), B_1 and B_2 are bridges of Y in $M(\Sigma)$ and by Lemma 5.2.8, there are $\overline{S_1} \in \pi(M(\Sigma), B_1, Y)$ such that $\overline{S_1} = (S_1 - z) \cup Y_2$ and $\overline{S_2} \in \pi(M(\Sigma), B_2, Y)$ such that $\overline{S_2} = S_2$. Thereby $\overline{S_1} \cup \overline{S_2} = Y$. Suppose that $z \in S_1 \cap S_2$, then $\overline{S_1} = (S_1 - z) \cup Y_2$ and $\overline{S_2} = (S_2 - z) \cup Y_2$ and, therefore, B_1, B_2 are avoiding bridges of Y in $M(\Sigma)$. Hence each class $\mathscr{U}_i^j$ consists of all-avoiding bridges of Y in $M(\Sigma)$.

Next we shall prove that the classes $\mathscr{U}_1^1$ and $\mathscr{U}_2^1$ can be merged into one class $\mathscr{U}_1$ consisting of all-avoiding bridges of Y in $M(\Sigma)$. Let B_1, B_2 be two avoiding bridges of $Y_1 \cup z$ in $M(\Sigma_1)$ that are contained in $\mathscr{U}_1^1$ and B_1', B_2' be two avoiding bridges of $Y_2 \cup z$ in $M(\Sigma_2)$ that are contained in $\mathscr{U}_2^1$. For B_1, B_2 there are $S_1 \in \pi(M(\Sigma_1), B_1, Y_1 \cup z)$ and $S_2 \in \pi(M(\Sigma_1), B_2, Y_1 \cup z)$ such that $S_1 \cup S_2 = (Y_1 \cup z)$

while for B_1', B_2' there are $S_1' \in \pi(M(\Sigma_2), B_1', Y_2 \cup z)$ and $S_2' \in \pi(M(\Sigma_2), B_2', Y_2 \cup z)$ such that $S_1' \cup S_2' = (Y_2 \cup z)$. We shall consider only the case where ($z \in S_1$ but $z \notin S_2$) and ($z \in S_1'$ but $z \notin S_2'$), since the others follow similarly. As above for B_1, B_2 there are $\overline{S_1} \in \pi(M(\Sigma), B_1, Y)$ where $\overline{S_1} = (S_1 - z) \cup Y_2$ and $\overline{S_2} \in \pi(M(\Sigma), B_2, Y)$ where $\overline{S_2} = S_2$ such that $\overline{S_1} \cup \overline{S_2} = Y$. Similarly for B_1', B_2', there are $\overline{S_1'} \in \pi(M(\Sigma), B_1', Y)$ where $\overline{S_1'} = (S_1' - z) \cup Y_1$ and $\overline{S_2'} \in \pi(M(\Sigma), B_2', Y)$ where $\overline{S_2'} = S_2'$ such that $\overline{S_1'} \cup \overline{S_2'} = Y$. Since B_1, B_1' are bridges of Y in $M(\Sigma)$, by Lemma 5.2.8, it follows that $\overline{S_1} \cup \overline{S_1'} = Y - W_{M(\Sigma)}(B_1, B_1')$, which implies that B_1' and B_1 are avoiding bridges of Y in $M(\Sigma)$. Moreover, by Lemma 5.2.7, there is $S_2'' \in \pi(M(\Sigma_1), B_2, Y_1 \cup z)$ such that $z \in S_2''$ and by Lemma 5.2.8, we have that $(S_2'' - z) \cup Y_2 \in \pi(M(\Sigma), B_2, Y)$. Similarly for B_2' there is $S_2''' \in \pi(M(\Sigma_2), B_2', Y_2 \cup z)$ such that $z \in S_2'''$ and therefore $(S_2''' - z) \cup Y_1 \in \pi(M(\Sigma), B_2', Y)$. Thus B_1, B_1', B_2 and B_2' are avoiding bridges of Y in $M(\Sigma)$ and the classes $\mathscr{U}_1^1$ and $\mathscr{U}_2^1$ can be merged into one class $\mathscr{U}_1$ of all-avoiding bridges of Y in $M(\Sigma)$. Similarly, the classes $\mathscr{U}_1^2$ and $\mathscr{U}_2^2$ can be merged into one class $\mathscr{U}_2$ consisting of all-avoiding bridges of Y in $M(\Sigma)$. $\square$

5.2.3 3-sum

The two types of 3-sum regarding signed graphs will be examined separately and structural results will be provided for the corresponding signed-graphic matroids. In contrast with the 1-sum and 2-sum operations where we were able to show how cocircuits and avoidance behave for general matroids, for the 3-sum operation we need to restrict ourselves to the class of signed-graphic matroids.

2-vertex 3-sum

We shall consider the case in which a connected signed graph Σ is decomposed to two connected signed graphs Σ_1 and Σ_2, where Σ is the 2-vertex 3-sum of Σ_1 and Σ_2. By the definition of the 2-vertex 3-sum operation, Σ has a 3-biseparation (X_1, X_2), where each signed graph $\Sigma|X_i$ $(i = 1, 2)$ is connected and unbalanced. Furthermore, each Σ_i is an unbalanced signed graph with $E(\Sigma_i) = X_i \cup Z$, where by Z the set of common edges of Σ_1 and Σ_2 inducing K_o is denoted. We shall also refer to Σ_i as the part of the 2-vertex 3-sum. Moreover, throughout this section, we shall denote by v_1 and v_2 the common vertices of $\Sigma|X_1$ and $\Sigma|X_2$ in Σ and by v_j^1 and v_j^2 $(j = 1, 2)$ the vertices in Σ_1 and Σ_2, respectively, which are identified so as to form v_j in Σ.

The following lemma presents the way a non-balancing bond of Σ is inherited to Σ_1 and Σ_2. Moreover, it establishes the relation between the separators of a non-

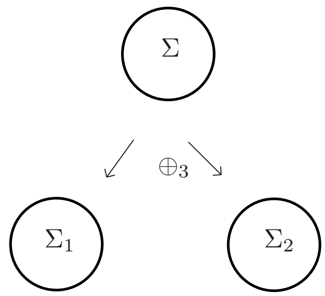


Figure 5.12: $M(\Sigma) = M(\Sigma_1) \oplus_3 M(\Sigma_2)$

balancing bond in a 2-vertex 3-sum signed graph $\Sigma = \Sigma_1 \oplus_3 \Sigma_2$ and the separators of the corresponding bond in the parts Σ_1 and Σ_2 of the 3-sum.

Lemma 5.2.12. *Let $M(\Sigma)$ be a connected signed-graphic matroid where Σ is the 2-vertex 3-sum of two signed graphs Σ_1 and Σ_2 with $M(\Sigma_1)$ and $M(\Sigma_2)$ connected and let $E(\Sigma_i) = X_i \cup Z$ $(i = 1, 2)$. If (Y_1, Y_2) is a partition of a non-balancing bond Y in Σ where $Y_i \subseteq E(\Sigma_i)$ then one of the following holds:*

(i) Y is a non-balancing bond in some Σ_i and the separators of $\Sigma_i \backslash Y$ are the separators of $\Sigma \backslash Y$ that are contained in $E(\Sigma_i)$ and one separator B with $Z \subseteq B$ such that the 2-vertex 3-sum $B \oplus_3 \Sigma_j (j \neq i)$ is a separator of $\Sigma \backslash Y$,

(ii) $Y_i \cup \overline{Z}$ is a bond in Σ_i, where $\overline{Z}$ contains every element z of Z such that $z \notin cl(X_i - Y_i)$ and the separators of $\Sigma_i \backslash Y_i \cup \overline{Z}$ are the separators of $\Sigma \backslash Y$ contained in $E(\Sigma_i)$ and a separator B_i such that $B_1 \oplus_2 B_2$ is a separator of $\Sigma \backslash Y$.

Proof. Since Y is a non-balancing bond in Σ, the signed graph $\Sigma \backslash Y$ consists of a balanced component, denoted by Σ^+ and one or more unbalanced components. As concerns the inheritance of Y to Σ_1 and Σ_2, we distinguish the following two cases, $Y \subseteq E(\Sigma_i)$ for some i or each Y_i is non-empty.

In the first case, assume that $Y \subseteq E(\Sigma_1)$. Due to the fact that Y is a minimal set of edges whose deletion increases the number of balanced components in Σ, the vertices v_1, v_2 must belong to the same unbalanced component of $\Sigma \backslash Y$. Since Σ_1 is part of the 2-vertex 3-sum, the vertices v_1^1, v_1^2 belong to the same unbalanced component of $\Sigma_1 \backslash Y$, denoted by Σ^-. Moreover, the signed graph $\Sigma_1 \backslash Y$ has the same connected components with $\Sigma \backslash Y$ apart from Σ^-, where the 2-vertex 3-sum of Σ^- and Σ_2 is the unbalanced component of $\Sigma \backslash Y$ containing v_1, v_2. Then the edges of Y in Σ_1 have an end-vertex at Σ^+ and the other at an unbalanced component of $\Sigma_1 \backslash Y$. Therefore Y is a minimal set of edges whose deletion increases the number of balanced components in Σ_1. Furthermore the existence of an unbalanced component in $\Sigma_1 \backslash Y$ implies that Y is a non-balancing bond of Σ_1. Since an unbalanced separator of $\Sigma \backslash Y$ contains both v_1 and v_2, there is an unbalanced separator B in $\Sigma_1 \backslash Y$ that contains the vertices v_1^1 and v_1^2 which are identified with the vertices v_2^1 and v_2^2 of Σ_2 so as to form v_1 and v_2 in Σ, respectively. Moreover, B contains all the edges of K_0. Then the unbalanced separator of $\Sigma \backslash Y$ that contains both v_1 and v_2 is a 2-vertex 3-sum of B and Σ_2. Furthermore the signed graph $\Sigma | X_i$ is isomorphic to $\Sigma_i | X_i$, therefore all separators of $\Sigma_1 \backslash Y$ that are contained in X_1 are separators of $\Sigma \backslash Y$ and (i) follows.

In the second case, each Y_i is non-empty, v_1 and v_2 cannot be vertices of the same unbalanced component of $\Sigma \backslash Y$. Thus, either one vertex in $\{v_1, v_2\}$ is a vertex of an unbalanced component of $\Sigma \backslash Y$ and the other is a vertex of Σ^+ or both v_1 and v_2 are vertices of Σ^+. In the first case, let us assume that v_1 is a vertex of an unbalanced component of $\Sigma \backslash Y$ and v_2 is a vertex of Σ^+. Since Y is a non-balancing bond in Σ, the edges of Y_i have an end-vertex at Σ^+ and one at an unbalanced component of $\Sigma \backslash Y$. Thereby the deletion of the edges of $Y_i \cup Z$ from Σ_i increases the number of balanced components by one and $\Sigma_i \backslash (Y_i \cup Z)$ has a unique balanced component, denoted by Σ_i^+. In Σ_i, the half-edge of K_0 at v_1 belongs to $cl(X_i - Y_i)$ so not in $\overline{Z}$. However, all the remaining edges of K_0 are contained in $\overline{Z}$ as they do not belong to $cl(X_i - Y_i)$. Hence in Σ_i, the elements of $\overline{Z}$ correspond to edges with an end-vertex at Σ_i^+. Then in Σ_i, each edge of Y_i has one end-vertex at Σ_i^+ and one at an unbalanced component of $\Sigma_i \backslash (Y_i \cup \overline{Z})$. Therefore $Y_i \cup \overline{Z}$ is a minimal set of edges whose deletion increases the number of balanced components in Σ_i. Since Σ_i is part of the 2-vertex 3-sum and $\Sigma | X_i$ is isomorphic to $\Sigma_i | X_i$, the separators of $\Sigma_i \backslash (Y_i \cup \overline{Z})$ that are contained in X_i are separators of $\Sigma \backslash Y$. Moreover, there is an unbalanced separator B_1 of $\Sigma_1 \backslash (Y_1 \cup \overline{Z})$ that contains $z \in Z \backslash \overline{Z}$ having v_1^1 as a vertex and an unbalanced separator B_2 of $\Sigma_2 \backslash (Y_2 \cup \overline{Z})$ that contains $z \in Z \backslash \overline{Z}$ having v_1^2 as a vertex. Thereby the 1-vertex 2-sum $B_1 \oplus_2 B_2$ is an unbalanced separator of $\Sigma \backslash Y$ containing v_1. Let us assume that both v_1 and v_2 are vertices of Σ^+. Since Σ_i is part of 2-vertex 3-sum, the deletion of the edges in $Y_i \cup Z$ from Σ_i increases the number of balanced components by one and let Σ_i^+ be the unique balanced component of $\Sigma_i \backslash (Y_i \cup Z)$. We shall consider only the case where the edges of Y_2 have both end-vertices at Σ_2^+ in Σ_2 and the edges of Y_1 have one end-vertex at Σ_1^+ and the other at some unbalanced component of $\Sigma_1 \backslash (Y_1 \cup Z)$ in Σ_1, since the case where the edges of Y_i have one end-vertex at Σ_i^+ and one at an unbalanced component of $\Sigma_i \backslash (Y_i \cup Z)$ follows similarly. In the signed graph Σ_i, the positive link of K_0 belongs to $cl(X_i - Y_i)$ so not in $\overline{Z}$ and the remaining edges of K_0 are contained in $\overline{Z}$ as they do not belong to $cl(X_i - Y_i)$. Then in Σ_i, the elements of $\overline{Z}$ correspond to edges that have both end-vertices at the balanced component of $\Sigma_i \backslash (Y_i \cup \overline{Z})$. Therefore, $Y_i \cup \overline{Z}$ is a bond in Σ_i and there is a balanced separator of $\Sigma_i \backslash (Y_i \cup \overline{Z})$, denoted by B_i, that contains the edge of K_0 that is not contained in $\overline{Z}$ and the vertices v_i^1 and v_i^2. Therefore, the 2-vertex 2-sum $B_1 \oplus_2 B_2$ constitutes Σ^+. Furthermore, since Σ_i is part of the 2-vertex 3-sum, each signed graph $\Sigma | X_i$ is isomorphic to $\Sigma_i | X_i$ and the separators of $\Sigma_i \backslash (Y_i \cup \overline{Z})$ that are contained in X_i are separators of $\Sigma \backslash Y$. □

Example 5.2.4. *Consider the signed graph Σ which is depicted in Figure 5.13. The signed graph Σ is 2-vertex 3-sum of Σ_1 and Σ_2 (Figure 5.12) where the signed graphs Σ_1 and Σ_2 are depicted in Figures 5.14(a) and 5.14(b) respectively. The signed-graphic matroid $M(\Sigma) = M(\Sigma_1) \oplus_3 M(\Sigma_2)$ has a non-graphic cocircuit $Y = \{8, 10, 11, -10\}$ which is bridge-separable and corresponds to a double bond in Σ. The bridges of Y in $M(\Sigma)$ are $B_1 = \{1, 2, 3, 4, 5, -1, -2, -3, -4, -5, -6, -7, 12\}$, $B_2 = \{6, 7, -8\}$, $B_3 = \{9\}$ and $B_4 = \{-9\}$ where $B_1 = B_1' \oplus z_1$. Then*

$$\begin{aligned}
\pi(M(\Sigma), B_1, Y) &= \{\{8\}, \{-10\}, \{10\}, \{11\}\} \\
\pi(M(\Sigma), B_2, Y) &= \{\{8\}, \{10, 11, -10\}\} \\
\pi(M(\Sigma), B_3, Y) &= \{\{10\}, \{11, 8, -10\}\} \\
\pi(M(\Sigma), B_4, Y) &= \{\{10\}, \{11\}, \{8, -10\}\}
\end{aligned}$$

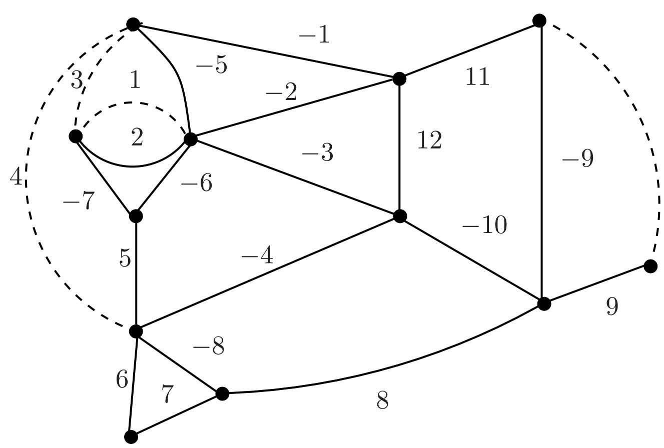
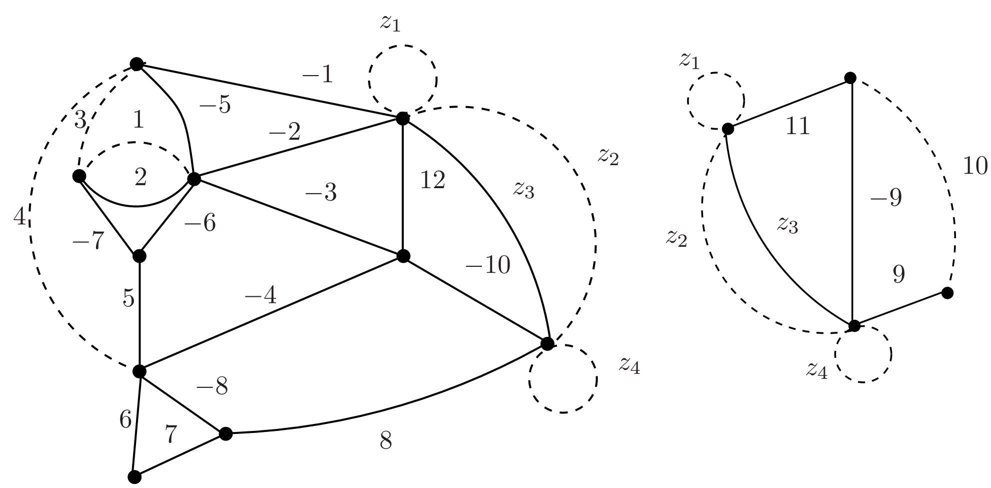


Figure 5.13: The signed graph Σ

Let (Y_1, Y_2) be a partition of Y such that $Y_1 = \{8, -10\}$ and $Y_2 = \{10, 11\}$ and $\overline{Z} = \{z_2, z_3, z_4\}$. Then $Y_1 \cup \overline{Z}$ is a bridge-separable cocircuit of $M(\Sigma_1)$ which corresponds to a double bond in Σ_1. Furthermore the bridges of $Y_1 \cup \overline{Z}$ in $M(\Sigma_1)$ are

$B_1' = \{1, 2, 3, 4, 5, -1, -2, -3, -4, -5, -6, -7, 12, z_1\}$*and* $B_2 = \{6, 7, -8\}$. *Then*

$$\pi(M(\Sigma_1), B_1', Y_1 \cup \overline{Z}) = \{\{8\}, \{-10\}, \{z_2\}, \{z_3\}, \{z_4\}\}$$
$$\pi(M(\Sigma_1), B_2, Y_1 \cup \overline{Z}) = \{\{8\}, \{z_2, z_3, z_4, -10\}\}.$$



(a) $Y_1 \cup z$ cocircuit of $M(\Sigma_1)$ (b) $Y_2 \cup z$ cocircuit of $M(\Sigma_2)$

Figure 5.14: The signed graphs Σ_1 (a) and Σ_2 (b)

Moreover, $Y_2 \cup \overline{Z}$ *is a bridge-separable cocircuit of* $M(\Sigma_2)$ *which corresponds to a double bond of* Σ_2. *The bridges of* Y *in* $M(\Sigma_2)$ *are* $B_1'' = \{z_1\}$, $B_3 = \{9\}$ *and* $B_4 = \{-9\}$. *Note that* B_1 *is 1-vertex 2-sum of* B_1' *and* B_1''. *Then*

$$\pi(M(\Sigma_2), B_1'', Y_2 \cup \overline{Z}) = \{\{11, z_3\}, \{10, z_4\}, \{z_2\}\}$$
$$\pi(M(\Sigma_2), B_3, Y_2 \cup \overline{Z}) = \{\{10\}, \{11, z_2, z_3, z_4\}\}$$
$$\pi(M(\Sigma_2), B_4, Y_2 \cup \overline{Z}) = \{\{10\}, \{11\}, \{z_2, z_3\}, \{z_4\}\}$$

3-vertex 3-sum

We shall now consider the case in which Σ is the 3-vertex 3-sum of two connected signed graphs Σ_1 and Σ_2. We shall refer to Σ_1 and Σ_2 as parts of the 3-vertex 3-sum. Then Σ has a 3-biseparation (X_1, X_2), where we shall assume that $\Sigma|X_1$ is the connected unbalanced signed graph and $\Sigma|X_2$ is the connected balanced signed graph. Moroever, throughout this section $V(\Sigma|X_1) \cap V(\Sigma|X_2) = \{v_1, v_2, v_3\}$ and $E(\Sigma_i) = X_i \cup Z$ $(i = 1, 2)$, where Z are the common edges of Σ_1 and Σ_2 inducing a positive triangle.

The following lemma describes all possible ways that a non-balancing bond of a signed graph can be inherited to the parts of a 3-vertex 3-sum. Moreover, it describes the way that the separators of a non-balancing bond in a signed graph that is a 3-vertex 3-sum of two signed graphs are inherited to the parts of the 3-vertex 3-sum.

Lemma 5.2.13. *Let $M(\Sigma)$ be a connected signed-graphic matroid where Σ is 3-vertex 3-sum of a connected unbalanced signed graph Σ_1 and a connected balanced signed graph Σ_2. If (Y_1, Y_2) is a partition of a non-balancing bond Y in Σ such that $Y_i \subseteq E(\Sigma_i), (i = 1, 2)$ then one of the following holds:*

(i) Y is a non-balancing bond in some Σ_i and its separators in $\Sigma_i \backslash Y$ are the separators of $\Sigma \backslash Y$ that are contained in $E(\Sigma_i)$ and a separator B with $Z \subseteq B$ such that the 3-vertex 3-sum $B \oplus_3 \Sigma_j, (j \neq i)$ is a separator in $\Sigma \backslash Y$,

(ii) $Y_i \cup \overline{Z}$ is a non-balancing bond in Σ_i, where $\overline{Z}$ contains every element z of Z such that $z \notin cl(X_i - Y_i)$, and its separators are the separators of $\Sigma \backslash Y$ contained in $E(\Sigma_i)$ and one separator B_i such that $B_1 \oplus_2 B_2$ is a separator of $\Sigma \backslash Y$.

Proof. Since Y is a non-balancing bond in Σ, the signed graph $\Sigma \backslash Y$ consists of a balanced component, denoted by Σ^+ and one or more unbalanced components. There are two possible cases that Y can be inherited to Σ_1 and Σ_2, $Y \subseteq E(\Sigma_i)$ for some i or each Y_i is non-empty.

In the first case, suppose that $Y \subseteq E(\Sigma_1)$. Since each Σ_i is part of the 3-vertex 3-sum, there is a connected component of $\Sigma \backslash Y$ that contains all three vertices of $\{v_1, v_2, v_3\}$ which is either balanced or unbalanced. Let us assume that the latter component is unbalanced, since the other case follows similarly. By definition of 3-vertex 3-sum, the signed graph $\Sigma_1 \backslash Y$ has the same components with $\Sigma \backslash Y$, apart from the unbalanced one that contains v_1, v_2, v_3 denoted by B. Moreover, $B \oplus_3 \Sigma_2$ constitutes the unique unbalanced component of $\Sigma \backslash Y$. Due to the fact that Y is a non-balancing bond in Σ, the deletion of Y from Σ_1 increases the number of balanced components and it is minimal with respect to this property. Furthermore there is an unbalanced component in $\Sigma_1 \backslash Y$ and therefore Y is a non-balancing bond in Σ_1. By definition of 3-vertex 3-sum, the separators of $\Sigma_1 \backslash Y$ are separators of $\Sigma \backslash Y$ except from a separator B such that $Z \subseteq B$, where the 3-vertex 3-sum of B and Σ_2 is the unique balanced component of $\Sigma \backslash Y$ and (i) follows.

In the second case, i.e., each Y_i is non-empty, v_1, v_2, v_3 cannot belong to the same component of $\Sigma \backslash Y$. Thereby either two vertices in $\{v_1, v_2, v_3\}$ belong to an unbalanced component of $\Sigma \backslash Y$ and the third to Σ^+ or two vertices in $\{v_1, v_2, v_3\}$ belong to Σ^+ and the third to an unbalanced component of $\Sigma \backslash Y$. In the first subcase, suppose that v_1, v_2 belong to an unbalanced component of $\Sigma \backslash Y$, denoted by Σ^-, and v_3 is a vertex of Σ^+. Moreover, let v_j^1 and v_j^2 $(j = 1, 2, 3)$ be vertices in Σ_1 and Σ_2, respectively which are identified to form v_j in Σ. Since Σ_i is part of the 3-vertex 3-sum, Σ^- is the 2-vertex 2-sum of Σ_1^- and Σ_2^-, where Σ_i^- denotes the signed graph which is obtained from $\Sigma^-|X_i$ by adding $z \in Z\backslash\overline{Z}$ as a positive link with end-vertices v_1^i and v_2^i. Furthermore Σ^+ is 1-sum of the signed graphs $\Sigma^+|X_1 = \Sigma_1^+$ and $\Sigma^+|X_2 = \Sigma_2^+$. Since Y is a non-balancing bond in Σ, the edges of Y_1 have one end-vertex at Σ_1^+ and the other at some unbalanced component of $\Sigma_1\backslash Z$, while the edges of Y_2 have one end-vertex at Σ_2^+ and the other at a balanced component of $\Sigma_2\backslash Z$. By definition of 3-vertex 3-sum, the edges of $\overline{Z}$ have v_3^i as a common end-vertex in Σ_i. Therefore $Y_i \cup \overline{Z}$ is a non-balancing bond in Σ_i. In the second subcase, where v_1, v_2 belong to Σ^+ and v_3 is a vertex of an unbalanced component of $\Sigma\backslash Y$, denoted by Σ^-, we replace Σ^- with Σ^+ and Σ^+ with Σ^- in the above case and it follows that $Y_i \cup \overline{Z}$ is a non-balancing bond in Σ_i. Then $\Sigma_i\backslash(Y_i \cup \overline{Z})$ consists of one balanced component and one or more unbalanced components. Since Σ_i is part of the 3-vertex 3-sum, Σ^- is the 2-vertex 2-sum of Σ_1^- and Σ_2^-, where Σ_i^- denotes the signed graph which is obtained from $\Sigma^-|X_i$ by adding $z \in Z\backslash\overline{Z}$ as a positive link with end-vertices v_1^i and v_2^i. Furthermore Σ^+ is 1-sum of the signed graphs $\Sigma^+|X_1$ and $\Sigma^+|X_2$. Thereby the separators of $\Sigma_i\backslash(Y_i \cup \overline{Z})$ are separators of $\Sigma\backslash Y$ apart from one B_i that contains $z \in Z\backslash\overline{Z}$ where $B_1 \oplus B_2$ is a separator of $\Sigma\backslash Y$ and (ii) follows. □

5.2.4 Avoidance and bridge-separability in 3-sum

The following three technical lemmas are needed in order to prove the main results of this section (Theorem 35 and 36) which are critical components in the proof of the decomposition theorem.

Lemma 5.2.14. *Let $M(\Sigma) = M(\Sigma_1) \oplus_3 M(\Sigma_2)$ be a connected signed-graphic matroid where $M(\Sigma_1)$ and $M(\Sigma_2)$ are connected signed-graphic matroids with $E(M(\Sigma_i)) = X_i \cup Z$ $(i = 1, 2)$. Moreover, let Y be a non-balancing bond in Σ. If Y is a non-balancing bond in some Σ_i and B is a separator in $\Sigma_i\backslash Y$ then either:*

(i) B contains no element of Z and $\pi(M(\Sigma_i), B, Y) = \pi(M(\Sigma), B, Y)$ or

(ii) $Z \subseteq B$ and $\pi(M(\Sigma_i), B, Y) = \pi(M(\Sigma), B \oplus_3 \Sigma_j, Y)$, $(j \neq i)$.

Proof. Suppose w.l.o.g. that $i = 1$. By Lemma 5.2.12(i), a separator in $\Sigma_1\backslash Y$ either contains no element of Z or it contains every element of Z. Let B_1 be a separator of $\Sigma_1\backslash Y$ that contains no element of Z and let B be the unbalanced separator that contains every element of Z. Then by Lemma 5.2.12(i), it follows that $\Sigma_1\backslash Y$ has the same separators with $\Sigma\backslash Y$ apart from B which is replaced by

$B \oplus_3 \Sigma_2$ in $\Sigma \backslash Y$. Therefore B_1 is a separator of $\Sigma \backslash Y$. Since $\Sigma.(B_1 \cup Y)|Y$ and $\Sigma_1.(B_1 \cup Y)|Y$ are obtained by contracting identical separators and the unbalanced separators $B \oplus_3 \Sigma_2$ and B in $\Sigma \backslash Y$ and $\Sigma_1 \backslash Y$, respectively, then these graphs (i.e. $\Sigma.(B_1 \cup Y)|Y$ and $\Sigma_1.(B_1 \cup Y)|Y$) are isomorphic. It follows that the family of bonds of $\Sigma.(B_1 \cup Y)|Y$ is equal to the family of bonds of $\Sigma_1.(B_1 \cup Y)|Y$ and $\pi(M(\Sigma_1), B_1, Y) = \pi(M(\Sigma), B_1, Y)$ by definition. The signed graphs $\Sigma_1.(B \cup Y)|Y$ and $\Sigma.(B \oplus_3 \Sigma_2 \cup Y)|Y$ are obtained by contracting the same separators in $\Sigma_1 \backslash Y$ and $\Sigma \backslash Y$, respectively, and then by deleting B and $B \oplus_3 \Sigma_2$ in the so-obtained signed graphs, respectively. Thereby $\Sigma_1.(B \cup Y)|Y$ is isomorphic to $\Sigma.(B \oplus_3 \Sigma_2 \cup Y)|Y$ and, thus, $\pi(M(\Sigma_1), B, Y) = \pi(M(\Sigma), B \oplus_3 \Sigma_2, Y)$. □

Lemma 5.2.15. *Let $M(\Sigma) = M(\Sigma_1) \oplus_3 M(\Sigma_2)$ be a connected signed-graphic matroid where $M(\Sigma_1)$ and $M(\Sigma_2)$ are connected signed-graphic matroids with $E(M(\Sigma_i)) = X_i \cup Z$ $(i = 1, 2)$. Suppose further that (Y_1, Y_2) is a partition of a non-balancing bond Y in Σ where $Y_i \subseteq E(M(\Sigma_i))$ and $Y_i \cap E(M(\Sigma_i)) \neq \emptyset$. If $Y_i \cup \overline{Z}$ is a bond in Σ_i, where $\overline{Z}$ contains every element z of Z such that $z \notin cl(X_i - Y_i)$ and B is a separator of $\Sigma_i \backslash (Y_i \cup \overline{Z})$ that contains no element of Z then there exists $S \in \pi(M(\Sigma), B, Y)$ such that $Y_j \subseteq S$ $(j \neq i)$.*

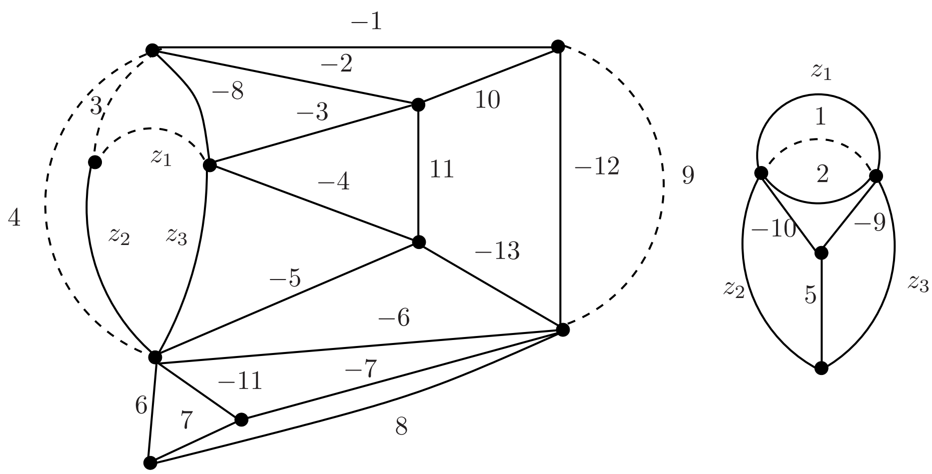
Proof. The proof is similar to that of Lemma 5.2.7. □

Example 5.2.5. *Consider the signed graph Σ which is depicted in Figure 5.9. The signed graph Σ is 3-vertex 3-sum of Σ_1 and Σ_2 (Figure 5.12) where the signed graphs Σ_1 and Σ_2 are depicted in Figures 5.15(a) and 5.15(b) respectively. The signed-graphic matroid $M(\Sigma) = M(\Sigma_1) \oplus_3 M(\Sigma_2)$ has a non-graphic cocircuit $Y = \{6, 9, -6, -11, -12, -13\}$ which is bridge-separable and corresponds to an unbalancing bond in Σ. The bridges of Y in $M(\Sigma)$ are $B_1 = \{1, 2, 3, 4, 5, 10, 11, -1, -2, -3, -4, -5, -8, -9, -10\}$ and $B_2 = \{7, 8, -7\}$. Then*

$$\pi(M(\Sigma), B_1, Y) = \{\{-6, -11, 6\}, \{-13\}, \{-12\}, \{9\}\}$$
$$\pi(M(\Sigma), B_2, Y) = \{\{-11\}, \{6\}, \{9, -6, -12, -13\}\}$$

The unbalancing bond Y, which is contained in $E(\Sigma_2)$, corresponds to an unbalancing bond in Σ_2. Moreover, the bridges of Y in $M(\Sigma)$ are $B_1' = \{z_1, 3, 4, 10, 11, -1, -2, -3, -4, -5, -8\}$ and $B_2 = \{7, 8, -7\}$. Then

$$\pi(M(\Sigma_1), B_1', Y_1 \cup \overline{Z}) = \{\{-6, -11, 6\}, \{-13\}, \{-12\}, \{9\}\}$$
$$\pi(M(\Sigma_1), B_2, Y_1 \cup \overline{Z}) = \{\{-11\}, \{6\}, \{9, -6, -12, -13\}\}.$$



(a) $Y_1 \cup z$ cocircuit of $M(\Sigma_1)$ (b) $Y_2 \cup z$ cocircuit of $M(\Sigma_2)$

Figure 5.15: The signed graphs Σ_1 (a) and Σ_2 (b)

Lemma 5.2.16. *Let $M(\Sigma)$ be a connected signed-graphic matroid where Σ is the l-vertex 3-sum $l = 2, 3$ of two signed graphs Σ_1 and Σ_2 with $M(\Sigma_1)$ and $M(\Sigma_2)$ connected and let $E(\Sigma_i) = X_i \cup Z$ $(i = 1, 2)$. Suppose further that (Y_1, Y_2) is a partition of a non-balancing bond Y in Σ where $Y_i \subseteq E(M(\Sigma_i))$ and $Y_i \cap E(M(\Sigma_i)) \neq \emptyset$. If B is a separator of $\Sigma\backslash Y$ and $S \in \pi(M(\Sigma), B, Y)$ then either:*

(i) B is a separator of $\Sigma_i\backslash(Y_i \cup \overline{Z})$ for some i and $S \in \pi(M(\Sigma_i), B, Y_i \cup \overline{Z})$, if $Y_j \nsubseteq S$ $(i \neq j)$, or $(S - Y_j) \cup \overline{Z} \in \pi(M(\Sigma_i), B, Y_i \cup \overline{Z})$, otherwise, or

(ii) $B = B_1 \oplus_2 B_2$ with B_i being a separator of $\Sigma_i\backslash(Y_i \cup \overline{Z})$ containing $z \in cl(X_i - Y_i)$ and $S \cup Z' \in \pi(M(\Sigma_i), B_i, Y_i \cup \overline{Z})$ $(j \neq i)$, where $Z' \subseteq \overline{Z}$ and $S \subseteq Y_i$.

Proof. The signed graph $\Sigma\backslash Y$ consists of a unique balanced component, denoted by Σ^+ and one or more unbalanced components. Let us assume first that $l = 2$. Since each Y_i is non-empty, the vertices v_1, v_2 cannot belong to the same unbalanced component of $\Sigma\backslash Y$. Thus either one vertex in $\{v_1, v_2\}$ is a vertex of an unbalanced component of $\Sigma\backslash Y$ and the other is a vertex of Σ^+ or both v_1 and v_2 are vertices of Σ^+. We consider only the first case and suppose that v_1 is a vertex of an unbalanced component of $\Sigma\backslash Y$ and v_2 is a vertex of Σ^+, since the case where both v_1, v_2 belong to Σ^+ is similar. Due to the fact that B is a separator of $\Sigma\backslash Y$, then by Lemma 5.2.12(ii), B is either a separator of $\Sigma_i\backslash(Y_i \cup \overline{Z})$ for some i or $B = B_1 \oplus_2 B_2$, where B_i is the separator of $\Sigma_i\backslash(Y_i \cup \overline{Z})$ that contains $z \in Z\backslash\overline{Z}$. Thus we distinguish the following cases for B:

<u>**Case 1:**</u> B is a separator of $\Sigma_i \backslash (Y_i \cup \overline{Z})$ for some i.
Suppose w.l.o.g. that $i = 1$ and moreover, let us suppose that B is a balanced separator of Σ^+. In Σ, the edges of Y_1 have either one end-vertex at Σ^+ and one at an unbalanced component of $\Sigma \backslash Y$ or both end-vertices at Σ^+ (or one end-vertex at Σ^+ if the edge is a joint). Thus they become joints at a vertex of attachment of B or classes of parallel edges of the same sign incident at two vertices of attachment of B in $\Sigma.(B \cup Y)|Y$. By Lemma 5.2.15, it follows that the edges of Y_2 are contained in a bond of $\Sigma.(B \cup Y)|Y$. Since Σ_1 is part of the 2-vertex 3-sum, each bond in $\Sigma.(B \cup Y)|Y$ is a bond in $\Sigma_1.(B \cup Y_1 \cup \overline{Z})|(Y_1 \cup \overline{Z})$, apart from the one that contains Y_2. Moreover, the bond which contains the edges of Y_2 in $\Sigma.(B \cup Y)|Y$ is transformed to a bond in $\Sigma_1.(B \cup Y_1 \cup \overline{Z})|(Y_1 \cup \overline{Z})$ by replacing the edges of Y_2 with the edges of $\overline{Z}$. Therefore, there is a one to one correspondence between the bonds of $\Sigma.(B \cup Y)|Y$ and the bonds of $\Sigma_1.(B \cup Y_1 \cup \overline{Z})|(Y_1 \cup \overline{Z})$. Each element in $\pi(M(\Sigma), B, Y)$ is a minimal nonempty intersection of bonds in $\Sigma.(B \cup Y)|Y$. Thereby if $S \in \pi(M(\Sigma), B, Y)$ does not contain Y_2, then $S \in \pi(M(\Sigma_1), B, Y_1 \cup \overline{Z})$, otherwise $(S - Y_2) \cup \overline{Z} \in \pi(M(\Sigma_1), B, Y_1 \cup \overline{Z})$. The case where B is an unbalanced separator of $\Sigma \backslash Y$ different from $B_1 \oplus_2 B_2$ or a balanced separator of an unbalanced component follow similarly.
<u>**Case 2:**</u> $B = B_1 \oplus_2 B_2$.
Then B is the unbalanced separator of $\Sigma \backslash Y$ having v_1 as a vertex. By definition the elements of $\pi(M(\Sigma), B, Y)$ partition Y and $S \in \pi(M(\Sigma), B, Y)$ is a minimal nonempty intersection of bonds in $\Sigma.(B \cup Y)|Y$. Since Σ_i is part of the 2-vertex 3-sum, each bond contained in X_i in $\Sigma.(B \cup Y)|Y$ is a bond in $\Sigma_i.(B_i \cup Y_i \cup \overline{Z})|(Y_i \cup \overline{Z})$. Thereby there is $S \cup Z'$, where $Z' \subseteq \overline{Z}$, that is a minimal nonempty intersection of bonds in $\Sigma_i.(B \cup Y_i \cup \overline{Z})|(Y_i \cup \overline{Z})$ and, therefore, $S \cup Z' \in \pi(M(\Sigma_i), B_i, Y_i \cup \overline{Z})$.

Let us assume now that $l = 3$, since each Y_i is nonempty, v_1, v_2, v_3 cannot belong to the same component of $\Sigma \backslash Y$. Thereby either two vertices in $\{v_1, v_2, v_3\}$ belong to an unbalanced component of $\Sigma \backslash Y$ and the third to Σ^+ or two vertices in $\{v_1, v_2, v_3\}$ belong to Σ^+ and the third to an unbalanced component of $\Sigma \backslash Y$. In the first case, suppose that v_1, v_2 belong to an unbalanced component of $\Sigma \backslash Y$ and v_3 is a vertex of Σ^+, while in the second case suppose that v_1, v_2 are vertices of Σ^+ and v_3 is a vertex of an unbalanced component of $\Sigma \backslash Y$ and the result follows as above. $\square$

From the above two lemmas, we deduce that when $Y_i \cup \overline{Z}$ is a bond in Σ_i and B is a separator of $\Sigma_i \backslash (Y_i \cup \overline{Z})$ that contains no element of Z, then there exists $S \in \pi(M(\Sigma_i), B, Y_i \cup \overline{Z})$ such that $\overline{Z} \subseteq S$. The following lemma presents the relation between the elements of $\pi(M(\Sigma_i), B_i, Y_i \cup \overline{Z})$ and $\pi(M(\Sigma), B'_i, Y)$, where

B_i is a bridge of $M(\Sigma_i)\backslash(Y_i \cup \overline{Z})$ and B'_i is the bridge of Y in $M(\Sigma)$ that contains an element of B_i.

Lemma 5.2.17. *Let $M(\Sigma)$ be a connected signed-graphic matroid where Σ is the l-vertex 3-sum $l = 2, 3$ of two signed graphs Σ_1 and Σ_2 with $M(\Sigma_1)$ and $M(\Sigma_2)$ connected and let $E(\Sigma_i) = X_i \cup Z$ $(i = 1, 2)$. Suppose further that (Y_1, Y_2) is a partition of a non-balancing bond Y in Σ where $Y_i \subseteq E(M(\Sigma_i))$ and $Y_i \cap E(M(\Sigma_i)) \neq \emptyset$. If B_1 is a separator of $\Sigma_1\backslash Y_1 \cup \overline{Z}$ and $S \in \pi(M(\Sigma_1), B_1, Y_1 \cup \overline{Z})$ then either:*

- *(i) B_1 is a separator of $\Sigma\backslash Y$ and $S \in \pi(M(\Sigma), B_1, Y)$, if $\overline{Z} \nsubseteq S$, or $(S-\overline{Z}) \cup Y_2 \in \pi(M(\Sigma), B_1, Y)$, otherwise, or*
- *(ii) $B = B_1 \oplus_2 B_2$ with B_i being a separator of $\Sigma_i\backslash(Y_i \cup \overline{Z})$ containing $z \in cl(X_i - Y_i)$ and $S - Z' \in \pi(M(\Sigma), B_i, Y)$ $(j \neq i)$, where $Z' \subseteq \overline{Z}$.*

Example 5.2.6. *Consider the signed graph Σ which is depicted in Figure 5.9. The signed graph Σ is 3-vertex 3-sum of Σ_1 and Σ_2 (Figure 5.12) where the signed graphs Σ_1 and Σ_2 are depicted in Figures 5.15(a) and 5.15(b) respectively. The signed-graphic matroid $M(\Sigma) = M(\Sigma_1) \oplus_3 M(\Sigma_2)$ has a non-graphic cocircuit $Y = \{8, 4, -6, -7, 5, -5\}$ which is bridge-separable and corresponds to an unbalancing bond in Σ. The bridges of Y in $M(\Sigma)$ are $B_2 = \{1, 2, 3, 9, 10, 11, -1, -2, -3, -4, -8, -9, -10, -12, -13\}$ and $B_1 = \{6, 7, -11\}$. Then*

$$\pi(M(\Sigma), B_1, Y) = \{\{-5, -6, 4, 5\}, \{-7\}, \{8\}\}$$
$$\pi(M(\Sigma), B_2, Y) = \{\{-5\}, \{5\}, \{4\}, \{-6, -7, 8\}\}$$

Let (Y_1, Y_2) be a partition of Y such that $Y_1 = \{8, -6, -7, -5, 4\}$ and $Y_2 = \{5\}$ and $\overline{Z} = \{z_2, z_3\}$. Then $Y_1 \cup \overline{Z}$ is a bridge-separable cocircuit of $M(\Sigma_1)$ which corresponds to an unbalancing bond in Σ_1. Furthermore the bridges of $Y_1 \cup \overline{Z}$ in $M(\Sigma_1)$ are $B'_2 = \{z_1, 3, 9, 10, 11, -1, -2, -3, -4, -8, -12, -13\}$ and $B_1 = \{6, 7, -11\}$. Then

$$\pi(M(\Sigma_1), B_1, Y_1 \cup \overline{Z}) = \{\{-5, -6, 4, z_2, z_3\}, \{-7\}, \{8\}\}$$
$$\pi(M(\Sigma_1), B'_2, Y_1 \cup \overline{Z}) = \{\{-5\}, \{z_2\}, \{4\}, \{-6, -7, 8\}, \{z_3\}\}.$$

Moreover, $Y_2 \cup \overline{Z} = \{5, z_2, z_3\}$ is a bridge-separable cocircuit of $M(\Sigma_2)$ which corresponds to an unbalancing bond of Σ_2. The bridges of $Y_2 \cup \overline{Z}$ in $M(\Sigma_2)$ are $B'_1 = \{1\}$ and $B''_2 = \{z_1, 2, -9, -10\}$. Note that $B_2 = B'_2 \oplus_2 B''_2$. Then

$$\pi(M(\Sigma_2), B'_1, Y_2 \cup \overline{Z}) = \{\{z_2, z_3, 5\}\}$$
$$\pi(M(\Sigma_2), B''_2, Y_2 \cup \overline{Z}) = \{\{z_2\}, \{z_3\}, \{5\}\}$$

The following theorem describes how the property of bridge-separability of a cocircuit in $M(\Sigma_1)$ or (and) $M(\Sigma_2)$ is inheritted to the induced cocircuit of the matroid $M(\Sigma) = M(\Sigma_1) \oplus_3 M(\Sigma_2)$.

Theorem 35. *Let $M(\Sigma) = M(\Sigma_1) \oplus_3 M(\Sigma_2)$ be a 3-connected signed-graphic matroid where $M(\Sigma_1)$ and $M(\Sigma_2)$ are 3-connected signed-graphic matroids with $E(M(\Sigma_i)) = X_i \cup Z$ $(i = 1, 2)$. Suppose further that (Y_1, Y_2) is a partition of a cocircuit Y of $M(\Sigma)$ such that $Y_i \subseteq E(\Sigma_i)$. If Y is a bridge-separable cocircuit of some $M(\Sigma_i)$ or $Y_i \cup \overline{Z}$ is a bridge-separable cocircuit of each $M(\Sigma_i)$, then Y is bridge-separable in $M(\Sigma)$.*

Proof. We distinguish the following two cases: either $Y \subseteq E(M(\Sigma_i))$ for some i or each Y_i is nonempty. In the first case, since Y is bridge-separable cocircuit in some $M(\Sigma_i)$, say $M(\Sigma_1)$, the bridges of Y in $M(\Sigma_1)$ can be partitioned into two classes where any two bridges of the same class are avoiding. By Lemma 5.2.12(i), each bridge of Y in $M(\Sigma_i)$, apart from one which contains the elements of Z, denoted by B, and $B \oplus_3 \Sigma_2$ is a bridge of $M(\Sigma)\backslash Y$. By Lemma 5.2.14, for each bridge B' of Y in $M(\Sigma_1)$ different form B, it holds that $\pi(M(\Sigma_1), B', Y) = \pi(M(\Sigma), B', Y)$, while for B it holds that $\pi(M(\Sigma_1), B, Y) = \pi(M(\Sigma), B \oplus_3 \Sigma_2, Y)$. Since Y is a bridge-separable cocircuit of $M(\Sigma_1)$, there is a partition of the bridges of Y in $M(\Sigma_1)$ into two classes where any two bridges in the same class are avoiding. Then Y is also bridge-separable cocircuit of $M(\Sigma)$. More precisely, the two classes of all-avoiding bridges of Y in $M(\Sigma)$ are the two classes of all-avoiding bridges of Y in $M(\Sigma_1)$, where B is replaced by $B \oplus_3 \Sigma_2$.

In the second case, by hypothesis, there is a partition of the bridges of $Y_i \cup \overline{Z}$ in each $M(\Sigma_i)$ into two classes $\mathscr{U}_i^1$ and $\mathscr{U}_i^2$ such that any two bridges in the same class are avoiding. Let B_1 and B_2 be two arbitrary bridges of some class $\mathscr{U}_i^j$ $(i, j = 1, 2)$, let $\mathscr{U}_1^1$, therefore there are $S_1 \in \pi(M(\Sigma_1), B_1, Y_1 \cup \overline{Z})$ and $S_2 \in \pi(M(\Sigma_1), B_2, Y_1 \cup \overline{Z})$ such that $S_1 \cup S_2 = Y_1 \cup \overline{Z}$. Then by Lemma 5.2.12(ii) and Lemma 5.2.13(ii), either B_1 and B_2 are bridges of Y in $M(\Sigma)$ or one of B_1 and B_2 contains $z \in cl(X_1 - Y_1)$, let B_1. In the first subcase, by avoidance of B_1, B_2 either $\overline{Z}$ belongs to at least one of S_1, S_2 or $\overline{Z} \subseteq S_1 \cap S_2$. Suppose first that $\overline{Z} \subseteq S_1$ but $\overline{Z} \nsubseteq S_2$. The case where $\overline{Z} \subseteq S_2$ but $\overline{Z} \nsubseteq S_1$ is similar. By Lemma 5.2.17(i), there are $\overline{S_1} \in \pi(M(\Sigma), B_1, Y)$ such that $\overline{S_1} = (S_1 - \overline{Z}) \cup Y_2$ and $\overline{S_2} \in \pi(M(\Sigma), B_2, Y)$ such that $\overline{S_2} = S_2$ and therefore $\overline{S_1} \cup \overline{S_2} = Y$. If $\overline{Z} \subseteq S_1 \cap S_2$, then there is $\overline{S_2} \in \pi(M(\Sigma), B_2, Y)$ such that $\overline{S_2} = (S_2 - \overline{Z}) \cup Y_2$ and since B_1, B_2 are avoiding bridges of $Y_1 \cup \overline{Z}$ in $M(\Sigma_1)$, it follows that B_1, B_2 are avoiding bridges of Y in $M(\Sigma)$. In the second subcase, let us assume that $B_1 \oplus_2 B$ is a bridge of Y in $M(\Sigma)$, where B is the bridge of $Y_2 \cup \overline{Z}$ in $M(\Sigma_2)$ that contains $z \in cl(X_2 - Y_2)$. Moreover, B_2 is a bridge of Y in $M(\Sigma)$ and by Lemma 5.2.15, there is $\overline{S_2} \in \pi(M(\Sigma), B_2, Y)$ such that $Y_2 \subseteq \overline{S_2}$.

By avoidance of B_1 and B_2 and Lemma 5.2.17, we have that $B_1 \oplus_2 B$ and B_2 are avoiding bridges of Y in $M(\Sigma)$. Therefore each class $\mathscr{U}_i^j$ consists of all-avoiding bridges of Y in $M(\Sigma)$. Let B'_1, B'_2 be two avoiding bridges of $Y_2 \cup \overline{Z}$ in $M(\Sigma_2)$ that are contained in $\mathscr{U}_1^2$. This implies that there are $S'_1 \in \pi(M(\Sigma_2), B'_1, Y_2 \cup \overline{Z})$ and $S'_2 \in \pi(M(\Sigma_2), B'_2, Y_2 \cup \overline{Z})$ such that $S'_1 \cup S'_2 = Y_2 \cup \overline{Z}$. Then either B'_1 and B'_2 are avoiding bridges of Y in $M(\Sigma)$ or one of B'_1 and B'_2, say B'_1, contains $z \in cl(X_2 - Y_2)$. In the first subcase, let us suppose that $\overline{Z} \subseteq S'_1$ but $\overline{Z} \nsubseteq S'_2$, since the other cases follow similarly. By Lemma 5.2.17(i), there are $\overline{S'_1} \in \pi(M(\Sigma), B'_1, Y)$ where $\overline{S'_1} = (S'_1 - \overline{Z}) \cup Y_1$ and $\overline{S'_2} \in \pi(M(\Sigma), B'_2, Y)$ where $\overline{S'_2} = S_2$ such that $\overline{S'_1} \cup \overline{S'_2} = Y$. In the second subcase, let us assume that $B_1 \oplus B'_1$ is a bridge of Y in $M(\Sigma)$ where B_1 contains $z \in cl(X_1 - Y_1)$. Then by avoidance of B'_1 and B'_2, we have that $B_1 \oplus B'_1$ and B'_2 are avoiding bridges of Y in $M(\Sigma)$. Since $\overline{S'_1} \cup \overline{S_1} = Y$, it follows that B_1 and B'_1 are avoiding bridges of Y in $M(\Sigma)$. Moreover, by Lemma 5.2.15, there are $S''_2 \in \pi(M(\Sigma), B_2, Y)$ such that $Y_2 \subseteq S''_2$ and $S'''_2 \in \pi(M(\Sigma), B'_2, Y)$ such that $Y_1 \subseteq S'''_2$ and therefore B_2, B'_2 are avoiding bridges of Y in $M(\Sigma)$. Thus B_1, B_2, B'_1, B'_2 and $B_1 \oplus B'_1$ are avoiding bridges of Y in $M(\Sigma)$ and the classes $\mathscr{U}_1^1$ and $\mathscr{U}_2^1$ can merged into one with all-avoiding bridges. Similarly $\mathscr{U}_1^2$ and $\mathscr{U}_2^2$ can merged into one class with all-avoiding bridges and therefore, Y is bridge-separable cocircuit of $M(\Sigma)$. □

The following theorem describes how the property of all-avoiding bridges of a cocircuit in $M(\Sigma) = M(\Sigma_1) \oplus_3 M(\Sigma_2)$ is inherited in the cocircuits of $M(\Sigma_1)$ and $M(\Sigma_2)$.

Theorem 36. *Let $M(\Sigma) = M(\Sigma_1) \oplus_3 M(\Sigma_2)$ be a connected signed-graphic matroid where $M(\Sigma_1)$ and $M(\Sigma_2)$ are connected signed-graphic matroids with $E(M(\Sigma_i)) = X_i \cup Z$ ($i = 1, 2$). Suppose further that (Y_1, Y_2) is a partition of a non-graphic cocircuit Y of $M(\Sigma)$ such that $Y_i \subseteq E(M(\Sigma_i))$. If Y has all-avoiding bridges then one of the following holds:*

(i) Y is a cocircuit that has all-avoiding bridges in some $M(\Sigma_i)$.

(ii) $Y_i \cup \overline{Z}$ is a cocircuit that has all-avoiding bridges in $M(\Sigma_i)$.

Proof. Since Y is a non-graphic cocircuit of $M(\Sigma)$, it is a non-balancing bond in Σ. We distinguish two cases as concerns the inheritance of Y to Σ_1 and Σ_2: $Y \subseteq E(M(\Sigma_i))$ for some i or each Y_i is non-empty. In the first case, suppose that $Y \subseteq E(M(\Sigma_1))$. Then by Lemma 5.2.12(i) and Lemma 5.2.13(i), Y is a cocircuit of $M(\Sigma_1)$. Let B_1, B_2 be two avoiding bridges of Y in $M(\Sigma)$ and B be the bridge of Y in $M(\Sigma_1)$ that contains Z. Assume first that the bridges B_1, B_2 are different from the bridge $B \oplus_3 \Sigma_2$; then by Lemma 5.2.12(i) and Lemma

5.2.13(i), B_1 and B_2 are bridges of Y in $M(\Sigma_1)$. By Lemma 5.2.14(i), we have that $\pi(M(\Sigma), B_j, Y) = \pi(M(\Sigma_1), B_j, Y)$ $(j = 1, 2)$ and, therefore, B_1 and B_2 are avoiding bridges of Y in $M(\Sigma_1)$. Suppose now that one of B_1 or B_2 is the bridge $B \oplus_3 \Sigma_2$, say B_1. Then, by Lemma 5.2.12(i), B_2 is a bridge of $M(\Sigma_1)\backslash Y$. Moreover by Lemma 5.2.14(ii), we have that B_1 and B_2 are avoiding bridges of Y in $M(\Sigma_1)$ and, therefore, Y has all-avoiding bridges in $M(\Sigma_1)$.

For (ii), let B_1 and B_2 be two avoiding bridges of Y in $M(\Sigma)$. Then there are $S_1 \in \pi(M(\Sigma), B_1, Y)$ and $S_2 \in \pi(M(\Sigma), B_2, Y)$ such that $S_1 \cup S_2 = Y$. Suppose first that $B_1, B_2 \subseteq E(M(\Sigma_i))$ and moreover, suppose w.l.o.g. that $i = 1$. By Lemma 5.2.12(ii) and Lemma 5.2.13(ii), $Y_1 \cup \overline{Z}$ is a cocircuit of $M(\Sigma_1)$ and B_1, B_2 are bridges of $Y_1 \cup \overline{Z}$ in $M(\Sigma_1)$ without any element of Z. Moreover, by Lemma 5.2.15, there is a set that contains Y_2 in each $\pi(M(\Sigma), B_i, Y)$ and by avoidance of B_1 and B_2, Y_2 is either contained in one of S_1, S_2 or $Y_2 \subseteq S_1 \cap S_2$. In the first case, let us assume that only S_1 contains Y_2. Then by Lemma 5.2.16(i), there are $S_1' \in \pi(M(\Sigma_1), B_1, Y_1 \cup \overline{Z})$ such that $S_1' = (S_1 - Y_2) \cup \overline{Z}$ and $S_2' \in \pi(M(\Sigma_1), B_2, Y_1 \cup \overline{Z})$ such that $S_2' = S_2$ and, therefore, $S_1' \cup S_2' = Y_1 \cup \overline{Z}$. In the second case, i.e. $Y_2 \subseteq S_1 \cap S_2$, by Lemma 5.2.16(i), there are $S_1' \in \pi(M(\Sigma_1), B_1, Y_1 \cup \overline{Z})$ such that $S_1' = (S_1 - Y_2) \cup \overline{Z}$ and $S_2' \in \pi(M(\Sigma_1), B_2, Y_1 \cup \overline{Z})$ such that $S_2' = (S_2 - Y_2) \cup \overline{Z}$ and, therefore, $S_1' \cup S_2' = Y_1 \cup \overline{Z}$. Suppose now that one of B_1, B_2, say $B_1 = B_1' \oplus_2 B_2'$ where B_1' and B_2' are bridges of $M(\Sigma_1)\backslash(Y_1 \cup \overline{Z})$ and $M(\Sigma_2)\backslash(Y_2 \cup \overline{Z})$, respectively containing both $z \in Z\backslash\overline{Z}$ and B_2 is contained in $E(M(\Sigma_1))$. Then by Lemma 5.2.15, there is a set that contains Y_2 in $\pi(M(\Sigma), B_2, Y)$. Furthermore by avoidance of B_1 and B_2, Y_2 is either contained in one of S_1, S_2 or in both of them. Let us assume that Y_2 is contained only in S_2, since the other case follows similarly. By Lemma 5.2.16, it follows that there are $S_1' \in \pi(M(\Sigma_1), B_1', Y_1 \cup \overline{Z})$ such that $S_1' = S_1 \cup Z'$ where $Z' \subseteq \overline{Z}$ and $S_2' \in \pi(M(\Sigma_1), B_2, Y_1 \cup \overline{Z})$ such that $S_2' = (S_2 - Y_2) \cup \overline{Z}$ and, therefore, $S_1' \cup S_2' = Y_1 \cup \overline{Z}$. $\square$

The inheritance of a star bond and a balancing bond via 3-sums is described in the following two results.

Lemma 5.2.18. *Let Y be a cocircuit with all-avoiding bridges of a signed-graphic matroid $M(\Sigma) = M(\Sigma_1) \oplus_3 M(\Sigma_2)$ where $M(\Sigma_i)$ $(i = 1, 2)$ connected signed-graphic matroid with $E(M(\Sigma_i)) = X_i \cup \overline{Z}$ and $Y \subseteq E(M(\Sigma_i))$ for some i.*

(i) If Y is the star of a vertex in Σ_1 or Σ_2, then Y is the star of a vertex in Σ.

(ii) If Σ is the 3-vertex 3-sum of Σ_1 and Σ_2 and Y is balancing bond in the unbalanced Σ_i then Y is balancing bond in Σ.

(iii) If Σ is the 2-vertex 3-sum of Σ_1 and Σ_2, then Y cannot be balancing bond in Σ_i for some i.

Proof. Suppose that $Y \subseteq E(M(\Sigma_1))$. Then by Lemmas 5.2.12 and 5.2.13, Y is a cocircuit of $M(\Sigma_1)$. For (i) by hypothesis, Y is the star of a vertex w in Σ_1. Assume first that Σ is the 3-vertex 3-sum of Σ_1 and Σ_2. Moreover, let v_1, v_2, v_3 be the common vertices of $\Sigma|X_1$ and $\Sigma|X_2$ in Σ and v_j^1, v_j^2, v_j^3 $(j = 1, 2)$ be the vertices in Σ_1 and Σ_2, respectively which are identified so as to form v_1, v_2, v_3 in Σ. Since Y is a bond in Σ, it follows that $\overline{Z} \not\subseteq Y$, therefore Y cannot be the star of a vertex in $\{v_1^1, v_1^2, v_1^3\}$ in Σ_1. By definition of 3-vertex 3-sum, Σ_2 is either balanced or unbalanced and in both cases, by Lemma 5.2.13(i), Y is the star of w in Σ. Assume now that Σ is the 2-vertex 3-sum of Σ_1 and Σ_2. Moreover, let v_1, v_2 be the common vertices of $\Sigma|X_1$ and $\Sigma|X_2$ in Σ and v_j^1, v_j^2 $(j = 1, 2)$ be the vertices in Σ_1 and Σ_2, respectively which are identified so as to form v_j in Σ. Since Y is a bond in Σ, it follows that $\overline{Z} \not\subseteq Y$, and therefore Y cannot be the star of v_1^1 or v_1^2 in Σ_1. Then by definition of 2-vertex 3-sum and Lemma 5.2.12(i), Y is the star of w in Σ. Assuming that Σ_1 and Σ_2 are both balanced, then Y is also the star of w in Σ.

For (ii), since Y is a balancing bond in Σ_1, then by the definition of a balancing bond, Σ_1 is an unbalanced signed graph, while Σ_2 is balanced. Since the edges of $\overline{Z}$ induce a positive triangle in Σ_i, by performing switchings at the vertices of Σ_1, all edges of $\Sigma_1 \backslash Y$ become positive. Furthermore, by minimality of Y its edges have a negative sign in Σ_1. Thus, by definition of 3-vertex 3-sum, Y is a balancing bond in Σ.

For (iii) by way of contradiction assume that Y is a balancing bond in Σ_1. Applying switchings at the vertices of Σ_1 all edges of $\Sigma_1 \backslash Y$ become positive. Then by minimality of Y its edges are the only negative edges of Σ_1. By definition of 2-vertex 3-sum of two signed graphs, $\Sigma_2 \backslash \overline{Z}$ is unbalanced. Therefore Y is not a bond in Σ, which is a contradiction since Y is a cocircuit in $M(\Sigma)$. □

Lemma 5.2.19. *Let Y be a cocircuit with all-avoiding bridges of a signed-graphic matroid $M(\Sigma) = M(\Sigma_1) \oplus_3 M(\Sigma_2)$ where $M(\Sigma_i)$ $(i = 1, 2)$ connected signed-graphic matroid with $E(M(\Sigma_i)) = X_i \cup Z$. Suppose further that (Y_1, Y_2) is a partition of Y where $Y_i \subseteq E(M(\Sigma_i))$ and $Y \cap E(M(\Sigma_i)) \neq \emptyset$.*

(i) If $Y_i \cup \overline{Z}$ is the star of a vertex in Σ_i, then Y is the star of a vertex in Σ.

(ii) If Σ is the 2-vertex 3-sum of Σ_1 and Σ_2 and $Y_i \cup \overline{Z}$ is a balancing bond in each Σ_i then Y is a balancing bond in Σ.

(iii) If Σ is the 3-vertex 3-sum of Σ_1 and Σ_2, then $Y_i \cup \overline{Z}$ cannot be a balancing bond in each Σ_i.

(iv) If Σ is the 2-vertex 3-sum of Σ_1 and Σ_2 and $Y_i \cup \overline{Z}$ is the star of a vertex in one of Σ_1, Σ_2 and a balancing bond in the other, then Y is a balancing bond in Σ.

(v) *If Σ is the 3-vertex 3-sum of Σ_1 and Σ_2, then $Y_i \cup \overline{Z}$ cannot be the star of a vertex in one of Σ_1, Σ_2 and a balancing bond in the other.*

Proof. Assume first that Σ is the 3-vertex 3-sum of Σ_1 and Σ_2. Let us denote by v_1, v_2, v_3 be the common vertices of $\Sigma|X_1$ and $\Sigma|X_2$ in Σ and by v_j^1, v_j^2, v_j^3 $(j = 1, 2)$ the vertices in Σ_1 and Σ_2, respectively which are identified so as to form v_1, v_2, v_3 in Σ. For (i) since $Y_i \cup \overline{Z}$ is the star of v_i^1 or v_i^2 or v_i^3, let v_i^1, in Σ_i, then by Lemma 5.2.13, Y is the star of v_1 in Σ. For (iii), let us assume on the contrary that each $Y_i \cup \overline{Z}$ is a balancing bond in each Σ_i. Then by the definition of a balancing bond, Σ_1 and Σ_2 are both unbalanced, which contradicts the definition of 3-vertex 3-sum of two signed graphs. For (v) suppose that $Y_i \cup \overline{Z}$ is the star of a vertex in one of Σ_1, Σ_2 and a balancing bond in the other. This implies that at least one of Σ_1, Σ_2 is unbalanced. Suppose that Σ_1 is an unbalanced and Σ_2 is a balanced signed graph. Then $Y_1 \cup \overline{Z}$ is a balancing bond in Σ_1 and $Y_2 \cup \overline{Z}$ is the star of v_2^1 or v_2^2 or v_2^3, say v_2^1 in Σ_2. Since the latter signed graph is balanced, the links of Y_2 which are incident to v_2^1 are positive in Σ_2. Therefore there is a contradiction to minimality of Y in Σ.

Assume now that Σ is the 2-vertex 3-sum of Σ_1 and Σ_2. Let us denote by v_1, v_2 the common vertices of $\Sigma|X_1$ and $\Sigma|X_2$ in Σ and by v_i^1, v_i^2 be the vertices of Σ_1, Σ_2, respectively which are identified creating the vertex v in Σ. For (i), by hypothesis, $Y_i \cup \overline{Z}$ is the star of v_i^1 or v_i^2, say v_i^1 in Σ_i. Since the edges of $\overline{Z}$ determine uniquely the vertex v_i^1 or v_i^2 in Σ_i and moreover, these edges are common for Σ_1 and Σ_2, it follows by Lemma 5.2.12 that Y is the star of v_1 in Σ. For (ii), since $Y_i \cup \overline{Z}$ is a balancing bond in Σ_i, we perform switchings at the vertices of Σ_i and all edges of $\Sigma_i \backslash (Y_i \cup \overline{Z})$ become positive. Then due to the fact that $Y_i \cup \overline{Z}$ is a balancing bond in Σ_i, Y is a balancing bond in Σ. For (iv) by the definitions of 2-vertex 3-sum and a balancing bond the result follows.

□

Chapter 6

Binary signed-graphic matroids

Signed-graphic matroids are representable over any field of characteristic other than 2 [74]. Combining the above with results in [37], we distinguish the following three cases for a signed-graphic matroid M in terms of representability: (i) if M is binary, then it is regular and therefore, representable over all fields (ii) if M is representable over $GF(4)$ but not over $GF(2)$, then it is representable over all fields except $GF(2)$ (iii) if M is not representable over $GF(4)$, then it is representable over all fields of characteristic other than 2.

In this chapter, we present structural results for binary signed-graphic matroids and their signed graphic representations as well as characterizations which lead to algorithms. In section 6.1, we characterize graphically circuits, cocircuits and bases of binary signed-graphic matroids and we determine structural properties of tangled signed graphs. In section 6.2, inspired by Fournier's characterization for graphic matroids [15], we present a characterization for cographic signed-graphic matroids with not all-graphic cocircuits. In section 6.3, we prove results for cycles in jointless and in tangled signed graphs. In the last section, we furnish a characterization for binary signed-graphic matroids and we provide two algorithms: the first one receives as input a binary non-graphic matroid and checks whether it is isomorphic to the signed-graphic matroid of a given jointless signed graph and the second is a recognition algorithm for the class of binary signed-graphic matroids.

6.1 Tangled signed graphs

Binary signed-graphic and non-graphic matroids are represented by tangled signed graphs, as shown in the following theorem from [55]. The tangled signed graph $-K_5$, whose signed-graphic matroid is the regular matroid R_{10}, is depicted in Figure 6.1.

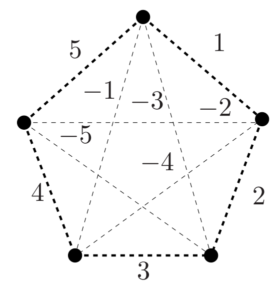


Figure 6.1: The signed graph $-K_5$

Theorem 37. *If Σ is connected and $M(\Sigma)$ is binary, then*

(i) Σ is tangled or

(ii) $M(\Sigma)$ is graphic.

Tangled signed graphs are unbalanced, have no balancing vertex and no two vertex-disjoint negative cycles. Hence we derive Propositions 21 and 22 which characterize graphically circuits and cocircuits of binary signed-graphic matroids. We note that a tangled signed graph is jointless.

Proposition 21. *If $M(\Sigma)$ is a connected binary signed-graphic and non-graphic matroid then it has no circuit which corresponds to a type II handcuff in the signed graph Σ.*

Proposition 22. *If Y is a cocircuit of a binary signed-graphic and non-graphic matroid $M(\Sigma)$, then Y corresponds to an unbalancing or a balancing bond in the signed graph Σ.*

Due to the structure of a tangled signed graph, the subgraph which is obtained by the deletion of an unbalancing bond consists of one unbalanced connected component. Moreover, the latter connected component contains exactly one unbalanced block.

Proposition 23. *If Y is a non-graphic cocircuit in a binary signed-graphic and non-graphic matroid $M(\Sigma)$, then the signed graph $\Sigma \backslash Y$ consists of one balanced and one unbalanced connected component.*

The relationship between the connectivity of a tangled signed graph and the associated binary signed-graphic matroid is given in the following result that appears in [56].

Theorem 38. *If Σ is a tangled signed graph without isolated vertices and $M(\Sigma)$ is k-connected for any $k \in \{2, 3\}$, then Σ is vertically k-connected.*

Every basis of a signed-graphic matroid $M(\Sigma)$ corresponds to a spanning 1-forest in the signed graph Σ by Theorem 5.1 of [74]. If B is a basis of a connected signed-graphic matroid $M(\Sigma)$, where Σ is an unbalanced signed graph, then the induced subgraph of B in Σ is either a spanning negative 1-forest or a spanning negative 1-tree. In case that $M(\Sigma)$ is binary, then B is a spanning negative 1-tree.

Proposition 24. *Every basis of a connected binary signed-graphic and non-graphic matroid $M(\Sigma)$ is a spanning negative 1-tree in Σ.*

Proof. $M(\Sigma)$ is a connected binary signed-graphic and non-graphic matroid, therefore, the signed graph Σ is connected and tangled. Let B denote a basis of $M(\Sigma)$ and $\Sigma[B]$ denote the subgraph of Σ induced by the edges of B. By way of contradiction suppose first that $\Sigma[B]$ is a spanning negative 1-forest i.e., all connected components of $\Sigma[B]$ are spanning negative 1-trees, then $\Sigma[B]$ contains two vertex-disjoint negative cycles. This implies that Σ has two vertex-disjoint negative cycles, which is a contradiction to tangleness of Σ. Let us assume that $\Sigma[B]$ is a spanning signed tree, since $\Sigma[B]$ is a balanced signed graph, it holds that $r(B) = r(\Sigma[B]) = v(\Sigma[B]) - 1 < r(\Sigma) = v(\Sigma[B])$ which is a contradiction. If we assume that $\Sigma[B]$ is either a spanning 1-forest or a signed forest then there is a contradiction to maximality of B. □

The following corollary is a straighforward consequence of Theorem 37 and Proposition 24.

Corollary 4. *If Σ is a connected tangled signed graph then every basis of $M(\Sigma)$ is a spanning negative 1-tree in Σ.*

Due to the structure of a tangled signed graph, the negative 1-paths with respect to a spanning negative 1-tree share a vertex.

Proposition 25. *If Σ is a connected tangled signed graph and T_Σ is a spanning negative 1-tree of Σ with negative cycle C_{T_Σ}, then the negative 1-paths (if there exist) with respect to T_Σ meet at a vertex w of C_{T_Σ}.*

Proof. Let P_{f_1}, P_{f_2} be two negative 1-paths with respect to T_Σ in Σ. All negative 1-paths have C_{T_Σ} as a common negative cycle by definition. Let C_1^-, C_2^- be the two negative cycles that are formed by the paths $P_{f_1} - C_{T_\Sigma}, P_{f_2} - C_{T_\Sigma}$ and the corresponding nonbasic edges f_1, f_2, respectively. Since Σ is tangled, any two negative cycles in Σ meet at a vertex. We shall show that C_1^-, C_2^- have the same common vertex $w \in V(C)$. Assume on the contrary that C_1^-, C_2^- have distinct common vertices w_1, w_2 with C_{T_Σ}, respectively. Since C_1^-, C_2^- are not vertex-disjoint, they have a common vertex w_3. Then there are two internally vertex-disjoint w_1, w_2-paths of C_{T_Σ}. Moreover, there are two vertex-disjoint paths w_1, w_3-path of $P_{f_1} - C_{T_\Sigma}$ and w_2, w_3 path of $P_{f_2} - C_{T_\Sigma}$. Thereby a cycle is formed with edges of T_Σ other than C_{T_Σ}, which is a contradiction. □

6.2 A characterization for cographic matroids with non-graphic cocircuits

Let C_1^*, C_2^*, C_3^* be three distinct cocircuits of a matroid M on a set E. We say that C_1^* *does not separate* C_2^* and C_3^* when $C_2^* \backslash C_1^*$ and $C_3^* \backslash C_1^*$ are included in the same connected component of $M \backslash C_1^*$. In [15], Fournier proved that a matroid is graphic if and only if for any three distinct cocircuits with a nonempty intersection, there exists one that separates the other two. In this subsection, inspired by Fournier's result, we characterize cographic signed-graphic matroids with not all-graphic cocircuits taking into advantage a structural property of cographic excluded minors of signed-graphic matroids.

The property of cocircuits, which is defined in the following, was used by Fournier in [15] in order to establish necessary and sufficient conditions for a matroid to be graphic.

Definition 6.2.1. *Let M be a matroid on E and let $A_1, A_2 \subset E$. We say that A_1 separates A_2 in M when A_2 meets at least two components of $M \backslash A_1$.*

The following result, which appears in [15], is used for the definition of a Fournier triple.

Proposition 26. *Let M be a matroid on E and $A_1, A_2 \subset Z \subset E$. If A_1 separates A_2 in M then A_1 separates A_2 in $M|Z$ and $M.Z$.*

We say that a cocircuit C_1^* *separates* a matroid M, namely C_1^* is a separating cocircuit, when C_1^* separates E in M. Furthermore, we say that a cocircuit C_1^* *separates two other cocircuits* C_2^* and C_3^* of a matroid M when C_1^* separates $C_2^* \cup C_3^*$ in M. Three cocircuits with a nonempty intersection such that none separates the other two is called a *Fournier triple*.

Examining all possible graphical representations of cocircuits of a Fournier triple in a binary signed-graphic and non-graphic matroid, we show that a Fournier triple cannot have two or three non-graphic cocircuits.

Lemma 6.2.1. *Let C_1^*, C_2^*, C_3^* be three cocircuits with a nonempty intersection of a connected binary signed-graphic and non-graphic matroid $M(\Sigma)$. If C_1^*, C_2^*, C_3^* are unbalancing bonds in Σ, then one of C_1^*, C_2^*, C_3^* separates the other two.*

Proof. Since $M(\Sigma)$ is a connected binary signed-graphic and non-graphic matroid, the signed graph Σ is connected and tangled. Moreover, $C_i^*, i = 1, 2, 3$ is an unbalancing bond in Σ and by Proposition 23, the signed graph $\Sigma \backslash C_i^*$ consists of one balanced component, denoted by B_i, and one unbalanced component, denoted by U_i. Let $e = \{b, v\}$ be the link of Σ such that $e \in C_1^* \cap C_2^* \cap C_3^*$ and suppose that $b \in B_i$ and $v \in U_i$. Assume on the contrary that (C_1^*, C_2^*, C_3^*) is a Fournier triple. Then by definition, $(C_2^* \cup C_3^*) - C_1^*$ is contained in a connected component of $\Sigma \backslash C_1^*$ which we may assume to be B_1, while the case of U_1 follows similarly. We distinguish two cases:
Case 1: $(C_1^* \cup C_3^*) - C_2^* \subseteq B_2$
The cocircuits C_1^* and C_2^* are distinct which implies that there exists $f \in C_2^* - C_1^*$ having an endvertex at U_2. Since U_2 contains no edge of $C_1^* \cup C_2^* \cup C_3^*$, the connected signed graph $U_2 \cup f$ is a subgraph of B_1 or U_1 in $\Sigma \backslash C_1^*$. Moreover, $f \in (C_2^* \cup C_3^*) - C_1^* \subseteq B_1$, therefore the unbalanced subgraph $U_2 \cup f$ is contained in the balanced B_1, which is a contradiction.
Case 2: $(C_1^* \cup C_3^*) - C_2^* \subseteq U_2$
Let us assume first that $(C_1^* \cup C_2^*) - C_3^* \subseteq B_3$. By the fact that C_1^* and C_3^* are distinct cocircuits, there exists $e_1 \in C_1^* - C_3^*$ with an endvertex in $V(U_1)$. Since U_1 contains no edge of $C_1^* \cup C_2^* \cup C_3^*$, the connected signed graph $U_1 \cup e_1$ is a subgraph of B_3 or U_3 in $\Sigma \backslash C_3^*$, which is a contradiction. Let us assume now that $(C_1^* \cup C_2^*) - C_3^* \subseteq U_3$. Then there exists $e_2 \in C_2^* - C_3^*$ with an endvertex at $V(B_2)$. Since $B_2 \cup e_2$ connected signed graph contained in B_3 or U_3 in $\Sigma \backslash C_3^*$, it holds that $B_2 \cup e_2 \subseteq U_3$. Thereby $V(B_2) \subseteq V(U_3)$ implying that $b \in V(U_3)$, which is a contradiction. $\square$

Lemma 6.2.2. *Let C_1^*, C_2^*, C_3^* be three cocircuits with a nonempty intersection of a connected binary signed-graphic and non-graphic matroid $M(\Sigma)$. If C_1^*, C_2^* are unbalancing bonds and C_3^* is a balancing bond in Σ, then one of C_1^*, C_2^*, C_3^* separates the other two.*

Proof. Since $M(\Sigma)$ is a connected binary signed-graphic and non-graphic matroid, the signed graph Σ is connected and tangled. By hypothesis $C_i^*, i = 1, 2$ is an unbalancing bond in Σ and by Proposition 23, the signed graph $\Sigma \backslash C_i^*$ consists of one balanced component, denoted by B_i, and one unbalanced component, denoted by U_i. In case of the balancing bond C_3^*, the signed graph $\Sigma \backslash C_3^*$ consists of one balanced component denoted by B_3. Let $e = \{b, v\}$ be the link of Σ such that

$e \in C_1^* \cap C_2^* \cap C_3^*$ and suppose that $b \in B_i$ and $v \in U_i$. Assume on the contrary that (C_1^*, C_2^*, C_3^*) is a Fournier triple. Then by definition, $(C_1^* \cup C_2^*) - C_3^*$ is contained in B_3. We distinguish two cases:

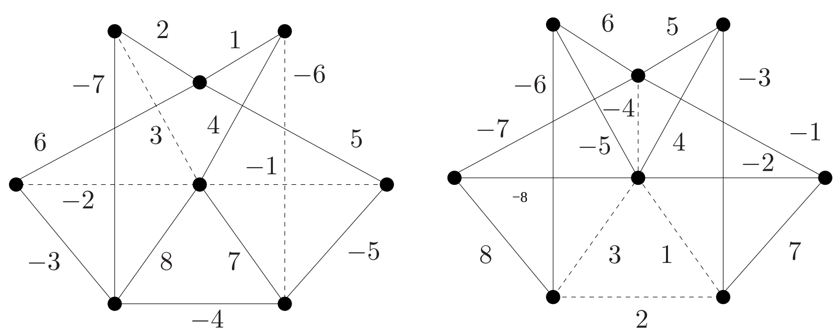
Case 1: $(C_1^* \cup C_3^*) - C_2^* \subseteq B_2$

Since C_2^*, C_3^* are distinct cocircuits, there is $h_2 \in C_2^* - C_3^*$ with an endvertex at U_2. Moreover, U_2 contains no edge in $C_1^* \cup C_2^* \cup C_3^*$, which implies that $U_2 \cup h_2$ is a connected subgraph contained in H_3 in $\Sigma \backslash C_3^*$.

Case 2: $(C_2^* \cup C_3^*) - C_1^* \subseteq U_2$

If $(C_2^* \cup C_3^*) - C_1^* \subseteq B_1$, then since C_2^*, C_3^* are distinct cocircuits, there is $h_1 \in C_1^* - C_3^*$ with an endvertex at U_1. Then $U_1 \cup h_1$ is a connected subgraph contained in H_3 in $\Sigma \backslash C_3^*$, which is a contradiction. Otherwise $(C_2^* \cup C_3^*) - C_1^* \subseteq U_1$. Since C_1^*, C_2^* are distinct cocircuits, there is $h_3 \in C_2^* - C_1^*$ with an endvertex at U_2. Thereby $B_2 \cup h_3$ is a connected subgraph contained in B_1 or U_1 in $\Sigma \backslash C_1^*$. Furthermore $h_3 \in U_1$ implying that $B_2 \cup h_3 \subseteq U_1$ which is a contradiction. □

Given a binary signed-graphic matroid, it is possible to have a Fournier triple where all three cocircuits are either balancing bonds or two of them are balancing bonds and one is an unbalancing bond. For the first case, consider as an example the signed graph Σ_{15} in Figure 6.2, which represents the dual matroid of R_{15}. All three nonseparating cocircuits of the Fournier triple (C_1^*, C_2^*, C_3^*) of R_{15}^*, where $C_1^* = \{-1, -3, -6, -7, 1\}$, $C_2^* = \{-2, -5, -6, -7, 2\}$ and $C_3^* = \{-2, -5, -6, 3\}$, are balancing bonds in Σ_{15}. For the second case, consider as an example the Fournier triple (C_1^*, C_4^*, C_5^*) of R_{15}^*, where $C_4^* = \{-1, -3, -7, 4\}$ is a balancing bond and $C_5^* = \{-1, -5, 5\}$ is a nonseparating and non-graphic cocircuit, which is an unbalancing bond in Σ_{15}.



(a) Σ_{15} represents R_{15}^* (b) Σ_{16} represents R_{16}^*

Figure 6.2: Signed graphic representations of R_{15}^* and R_{16}^*

The class of signed-graphic matroids has not been characterized yet in terms of excluded minors, however, the regular excluded minors of signed-graphic matroids were provided by Slilaty in [45]. The cographic matroids of the 29 graphs $G_1, \ldots, G_{29}$, which are excluded minors for the class of projective-planar graphs, are among the regular excluded minors of signed-graphic matroids. The 29 vertically 2-connected graphs $G_1, \ldots, G_{29}$ along with a representation matrix over $GF(2)$ are given in the Appendix of [41]. Moreover, a representation matrix over $GF(3)$ for each of R_{15} and R_{16} is given in Appendix 8.

Proposition 27. *A regular matroid is signed-graphic if and only if it has no minor isomorphic to $M^*(G_1), \ldots, M^*(G_{29}), R_{15}$ and R_{16}.*

Each cographic excluded minor of signed-graphic matroids contains a Fournier triple and a non-graphic cocircuit ([41] Appendix). A representation matrix over $GF(2)$ together with a Fournier triple with two non-graphic cocircuits are provided for each cographic excluded minor of signed-graphic matroids in Appendix 8. Thereby, we prove the following lemma.

Lemma 6.2.3. *Each cographic excluded minor of regular signed-graphic matroids with not all-graphic cocircuits has a Fournier triple with two non-graphic cocircuits.*

The following result is from [15].

Lemma 6.2.4. *If a minor $N = (M|B).A$ of a matroid M on E where $(A \subset B \subset E)$ possesses three cocircuits with a nonempty intersection such that none separates N, then M possesses three cocircuits with a nonempty intersection such that none separates the other two in M.*

The following lemma is a direct consequence of Lemma 6.2.4.

Lemma 6.2.5. *Let $N = (M|B).A$ be a minor of a matroid M on E where $(A \subset B \subset E)$, if N has a Fournier triple, then M has a Fournier triple.*

The following lemma, which appears in [41], implies that the all-graphic cociruits property is closed under minors ([41] Corollary 1). Thereby if a minor N of a matroid M has a non-graphic cocircuit, then M has a non-graphic cocircuit.

Lemma 6.2.6. *If N is a minor of a matroid M then for any cocircuit C_N of N there exists a cocircuit C_M of M such that $N \backslash C_N$ is a minor of $M \backslash C_M$.*

The following theorem characterizes the class of cographic and signed-graphic matroids with no minor isomorphic to $M^*(G_{17})$ and $M^*(G_{19})$. We note that the matroids R_{15} and R_{16} are not cographic ([45] Proposition 4.5).

Theorem 39. *Let M be a cographic matroid with no minor isomorphic to $M^*(G_{17})$ and $M^*(G_{19})$, then M is signed-graphic if and only if every Fournier triple has at most one non-graphic cocircuit.*

Proof. Let us assume first that M is a cographic and signed-graphic matroid. Then M is binary and by Lemmas 6.2.1 and 6.2.2 we have that every Fournier triple of M has at most one non-graphic cocircuit.

Let us assume now that M is not signed-graphic, therefore it has a minor M' which is isomorphic to a matroid in $\{M^*(G_1), \ldots, M^*(G_{17}), M^*(G_{19}), \ldots, M^*(G_{29})\}$. By Lemma 6.2.3, each of the matroids in $\{M^*(G_1), \ldots, M^*(G_{17}), M^*(G_{19}), \ldots, M^*(G_{29})\}$ have a Fournier triple with two non-graphic cocircuits. Then by Lemmas 6.2.6 and 6.2.5, it follows that M has a Fournier triple with two non-graphic cocircuits. □

6.3 Cycles in tangled signed graphs

Let B be a basis of a connected signed-graphic matroid $M(\Sigma)$ such that the signed graph Σ is jointless. Furthermore suppose that T_Σ is the spanning negative 1-tree of Σ with negative cycle C_{T_Σ} such that $E(T_\Sigma) = B$. The subgraphs of Σ which are induced by the sets $(P_f - C_{T_\Sigma}) \cup \{f\}$ for each $f \in E(M(\Sigma))$ are called *basic cycles* with respect to T_Σ. Every cycle in a connected and jointless signed graph can be expressed as symmetric difference of basic cycles with respect to the associate negative 1-tree.

Lemma 6.3.1. *If T_Σ is a spanning negative 1-tree of a connected and jointless signed graph $\Sigma = (G, \sigma)$, then every cycle of Σ is either a basic cycle or symmetric difference of basic cycles with respect to T_Σ.*

Proof. Consider a cycle C^- of Σ. If the cycle C^- is a basic cycle of Σ with respect to T_Σ, then the result follows. Thus, suppose that C^- is not a basic cycle. Then the spanning tree T_G, which is constructed from the negative 1-tree T_Σ by deleting an arbitrary edge e from the negative cycle C_T of T_Σ, constitutes a basis for the matroid $M(G)$. It follows that e is a nonbasic element of $M(G)$ with respect to the basis that is equal to T_G and therefore C_T is a fundamental cycle of G with respect to T_G. Since every cycle of Σ is also a cycle of G, it follows that C^- is symmetric difference of fundamental cycles of G with respect to T_G. It suffices to show that every fundamental cycle of G with respect to T_G is either a basic cycle or symmetric difference of basic cycles of Σ with respect to T_Σ. If there is no path $(P_f : f \in E(M) - B)$ such that $e \in P_f$, then all the fundamental cycles of G are basic by definition. Otherwise there is a path $(P_f : f \in E(M) - B)$ such that $e \in P_f$, and thereby C_T and $(P_f \cup \{f\}) \triangle C_T$ are fundamental cycles of G with respect to T_G. Since this holds for every path P_f such that $e \in P_f$, every fundamental cycle of G with respect to T_G is a basic cycle or symmetric difference of basic cycles of Σ with respect to T_Σ. □

Let C be a cycle and T_Σ be a negative 1-tree of a connected unbalanced signed graph Σ. If C is expressed as symmetric difference of basic cycles with respect to T_Σ by Lemma 6.3.1, then we shall refer to the aforementioned basic cycles as *basic cycles of* C with respect to T_Σ. If C is negative then the number of negative basic cycles of C with respect to T_Σ is odd.

Lemma 6.3.2. *Let T_Σ be a spanning negative 1-tree of a connected jointless signed graph $\Sigma = (G, \sigma)$. If C is a negative cycle of Σ, then the number of negative basic cycles of C with respect to T_Σ is odd.*

Proof. The negative edges of the basic cycles of C are partitioned into two sets: one set that contains the negative edges of C and one set that contains the negative edges that do not belong to C. Since C is a negative cycle of Σ, it contains an odd number of negative edges. Moreover, every negative edge of C belongs to an odd number of basic cycles of C, due to the definition of symmetric difference. Thus, the number of times that the negative edges of C appear in the basic cycles of C, denoted by S_1, is odd. The negative edges of the second set do not appear in C because each of them belongs to an even number of basic cycles of C with respect to T_Σ. Thereby the number of times they appear in the basic cycles of C, denoted by S_2, is even independently of their number. Then, the number of the negative edges in the basic cycles of C is equal to the sum of S_1 and S_2 and, therefore, it is odd. Equivalently, the total number of negative edges in the basic cycles of C is equal to the sum of the negative edges of the positive basic cycles of C and the negative edges of the negative basic cycles of C. Since the first part of the sum is even, then the second part is odd. It follows that the number of negative basic cycles of C with respect to T_Σ is odd, since every negative basic cycle has an odd number of negative edges. $\square$

A link which joins two vertices of a cycle in a graph but is not itself a link of the cycle is a *chord* of that cycle. A chord of a negative cycle is called a *minus-chord*. An edge of a signed graph which is neither a minus-chord nor a joint is called *m-edge*.

If we delete or contract a link e from a signed graph Σ, then the signed graphs $\Sigma \backslash e$ and Σ / e have no more negative cycles than Σ. Moreover, among the operations for taking minors, only the deletion of a link or the contraction of a chord from a negative cycle C of Σ may result in a signed graph with more negative cycles than Σ. Thereby the contraction of an m-edge is among the operations that leave the number of negative cycles unchanged when applied to a signed graph.

Lemma 6.3.3. *If e is a m-edge in a signed graph Σ then Σ / e has the same number of negative cycles with Σ.*

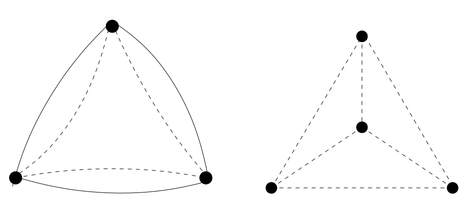
Proof. We shall show that there is one to one correspondence between the negative cycles of Σ and the negative cycles of Σ/e. Let us assume that the endvertices of $e = \{w_1, w_2\}$ are identified to a vertex w in Σ/e. A negative cycle that contains both endvertices of e in Σ, and therefore e, is mapped to the negative cycle in Σ/e that contains w and includes the same links apart from e. We note that the two cycles are both negative since the switching, performed when contracting e, leaves the sign of the cycle unchanged. A negative cycle that contains no endvertex of e in Σ is mapped to the identical negative cycle in Σ/e. A negative cycle of Σ that contains exactly one endvertex, say w_1 of e is mapped to the negative cycle of Σ/e that contains w instead of w_1 and the same set of links. When contracting a m-edge in Σ, neither a new negative cycle is formed nor a negative cycle is destroyed. Thus, there is a bijection between the negative cycles of Σ and the negative cycles of Σ/e. □

The following corollary derives easily by combining Lemma 6.3.3 with the definition of a tangled signed graph and the definition of a m-edge.

Corollary 5. *If e is a m-edge of a tangled signed graph Σ then Σ/e is tangled.*

A *minor* of a signed graph Σ is a signed graph obtained from Σ by performing the following operations: (1) contractions of edges, (2) deletions of edges, (3) switchings and (4) deletion of isolated vertices. A *link minor* is a minor which is obtained from Σ without performing contractions of joints. The two link minors of tangled signed graphs, which were provided by Slilaty in [55], are depicted in Figure 6.3.

Theorem 40 ([55] Theorem 3.16)**.** *If Σ is a tangled signed graph then Σ contains $-K_4$ or $\pm C_3$ as a link minor.*



(a) $\pm C_3$ (b) $-K_4$

Figure 6.3: Link minors of tangled signed graphs

Any signed graph obtained from a signed graph Σ by the addition of any number of parallel positive or negative links to existing links of Σ is said to belong to the parallel class of Σ, denoted by $\mathcal{P}(\Sigma)$.

Lemma 6.3.4. *If Σ is a tangled signed graph then there is a sequence of m-edge contractions which results in a signed graph belonging to either $\mathcal{P}(-K_4)$ or $\mathcal{P}(\pm C_3)$.*

Proof. By Theorem 40, there is a sequence of contractions of links and deletions of links and/or joints that reduces Σ to a signed graph, which up to switchings, is isomorphic to $-K_4$ or $\pm C_3$. Since deletions and contractions of links may be performed in any order, we first apply the set of contractions (of links) in Σ. Suppose now that at some point a chord of a negative cycle had to be contracted, then a negative loop (joint) is created which, after performing all contractions, should be attached at one of the vertices of the signed graph so-obtained, say Σ_c. According to Theorem 40, Σ_c is a graph belonging to $\mathcal{P}(-K_4)$ or $\mathcal{P}(\pm C_3)$ along with one or more joints at its vertices. Therefore, Σ_c is not tangled since it has two vertex disjoint negative cycles which, by Corollary 5, implies that Σ is not tangled; a contradiction. Therefore, no joints are created during the contractions and thus, no deletion of joints could take place, and the contractions applied are only m-edge contractions. Thus, regarding deletions, only deletion of links which are parallel to edges of $-K_4$ or $\pm C_3$ can be performed that is Σ_c is, up to switching, isomorphic to a graph belonging to either $\mathcal{P}(-K_4)$ or $\mathcal{P}(\pm C_3)$. □

A tangled signed graph Σ can be reduced by a sequence of m-edge contractions to a signed graph Σ' belonging to either $\mathcal{P}(-K_4)$ or $\mathcal{P}(\pm C_3)$ by Lemma 6.3.4. Moreover, by Lemma 6.3.3 the number of negative cycles of Σ equals the number of negative cycles of Σ'. Since the number of negative cycles of Σ' is polynomially bounded by the number of negative cycles of a signed graph belonging to either $\mathcal{P}(-K_4)$ or $\mathcal{P}(\pm C_3)$, it follows that the number of negative cycles of Σ is polynomially bounded. For a proof of the above result, we provide a case analysis in Appendix 8.

Theorem 41. *The number of negative cycles of a tangled signed graph is polynomially bounded by the number of negative cycles of a signed graph belonging to either $\mathcal{P}(-K_4)$ or $\mathcal{P}(\pm C_3)$.*

6.4 A characterization for binary signed-graphic matroids

In this subsection, we obtain a characterization for binary signed-graphic matroids and we derive representation matrices for signed-graphic matroids which are known to be $GF(3)$-representable. Total unimodularity is strongly connected with the representation matrices of binary signed-graphic matroids.

Let M be a matroid represented by a totally unimodular matrix D over $\mathbb{R}$. If D is viewed to be a matrix over an arbitrary field $\mathbb{F}$, then M is represented by D over $\mathbb{F}$. Furthermore, total unimodularity is maintained when we apply the following operations to a totally unimodular matrix: (1) pivots (2) row and column interchanges (3) scalings of rows or columns by -1. The *binary support* of a matrix D, denoted by $B_S(D)$, is defined to be a matrix which is obtained from D by replacing each non-zero entry of D by a 1.

The incidence matrix A_Σ of a signed graph Σ is a representation matrix of the signed-graphic matroid $M(\Sigma)$ over $GF(3)$ [44]. In the following proposition, we use the incidence matrix of a signed graph to highlight the connection between regular signed-graphic matroids and totally unimodular matrices.

Lemma 6.4.1. *Let $B = \{e_1, e_2, \ldots, e_r\}$ be a basis of a connected signed-graphic and non-graphic matroid $M(\Sigma)$, then $M(\Sigma)$ is regular if and only if the incidence matrix of the signed graph Σ is totally unimodular.*

Proof. Let us assume first that $M(\Sigma)$ is a regular matroid, then by definition there is a totally unimodular matrix A such that $M(\Sigma) \cong M[A]$. By a sequence of pivots, row and column interchanges, A can be transformed into a totally unimodular matrix of the form $[I_r|D_1]$, where the first r columns are labelled $e_1, e_2, \ldots, e_r$. Therefore $[I_r|D_1]$ has $\{0, \pm 1\}$ entries and it represents $M(\Sigma)$ over $GF(3)$. Since the incidence matrix A_Σ of Σ is over $GF(3)$, by applying pivots, row and column scalings to A_Σ, it can be transformed into a matrix $[I_r|D_2]$ in which the first r columns are labelled $e_1, e_2, \ldots, e_r$. By Proposition 6.4.1 of [35], it follows that $B_S(D_1) = B_S(D_2)$. Moreover, combining the fact that D_1 is totally unimodular with Proposition 10.2.4 of [35], we have that D_2 is totally unimodular.

For the converse, the incidence matrix A_Σ of Σ is totally unimodular and therefore, it is a representation matrix of $M(\Sigma)$ over $\mathbb{R}$. $\square$

Corollary 6. *A signed-graphic matroid $M(\Sigma)$ is binary if and only if the incidence matrix of the signed graph Σ is totally unimodular.*

The following well-known result, which appears in [17], characterizes the class of graphic matroids. This result is generalized for the class of binary signed-graphic and non-graphic matroids in Theorem 43.

Theorem 42. *If $\mathcal{B}$ is a basis of a binary matroid M, then M is graphic if and only if there is a tree T with $E(T) = \mathcal{B}$ such that each of the sets $(P_f : f \in E(M) - \mathcal{B})$ is a path in T.*

The following theorem characterizes the class of binary signed-graphic and non-graphic matroids. The main tools for its proof are structural and representation results for tangled signed graphs and the associated signed-graphic matroids proved in previous sections.

Theorem 43. *Let $\mathcal{B}$ be a basis of a connected binary and non-graphic matroid M, then M is signed-graphic if and only if*

(i) there is a negative 1-tree T with negative cycle C_T, that is not a half-edge and $E(T) = \mathcal{B}$ such that each of the sets $(P_f : f \in E(M) - \mathcal{B})$ is either a path or a negative 1-path in T

(ii) the signed graph obtained from T by adding each $f \in E(M) - \mathcal{B}$ as a link with endvertices the ends of P_f (resp. $P_f - C_T$) if P_f is a path (resp. negative 1-path) and with the same sign with P_f, has an incidence matrix that is totally unimodular.

Proof. For the "if" part. Since M is a connected binary signed-graphic and non-graphic matroid, there is a connected and tangled signed-graph Σ such that $M = M(\Sigma)$ [[55], Theorem 1.4]. By Proposition 24, there is a negative 1-tree T_Σ in Σ such that $E(T_\Sigma) = \mathcal{B}$ with negative cycle C_{T_Σ} that is not a half-edge. Moreover, by Proposition 21, the fundamental circuits of $M(\Sigma)$ with respect to $\mathcal{B}$ are either positive cycles or type I handcuffs in Σ. Therefore each set $(P_f : f \in E(\Sigma) - T_\Sigma)$ is either a path or a negative 1-path in T_Σ. Furthermore the incidence matrix of Σ is totally unimodular by Proposition 6.4.1, since $M(\Sigma)$ is regular.

For the "only if" part. Let us assume that there is a negative 1-tree T with negative cycle C_T that is not a joint and it holds that $E(T) = \mathcal{B}$. Moreover, for each $f \in E(M) - E(T)$ the set P_f is either a path or a negative 1-path in T. We construct a signed graph Σ from T by adding each $f \in E(M) - E(T)$ to T in the following manner. Suppose that P_f is a path in T, then we add edge f to T as a link with endvertices the two ends of P_f. Furthermore we attribute to f the sign of the path P_f. Therefore $P_f \cup \{f\}$ is a positive cycle in $T \cup \{f\}$. Suppose that P_f is a negative 1-path in T, then we add edge f to T as a link with endvertices the two ends of the path $P_f - C_T$. Furthermore we assign to f the opposite sign of the path $P_f - C_T$. Thereby $(P_f - C_T) \cup \{f\}$ is a negative cycle and $P_f \cup \{f\}$ is a type I handcuff in $T \cup \{f\}$. By assumption the incidence matrix of Σ is totally unimodular , which implies that $M(\Sigma)$ is regular and therefore, $M(\Sigma)$ is binary. Since both of M and $M(\Sigma)$ are binary and the fundamental circuits of $M(\Sigma)$ and M coincide with respect to the basis $\mathcal{B}$, it follows that $M \cong M(\Sigma)$. □

On combining Theorem 42 and Theorem 43, we derive a characterization for the class of binary signed-graphic matroids. Moreover, from the above theorems we obtain the following algorithm which determines whether a binary non-graphic matroid M given by an independence oracle is isomorphic to the signed-graphic matroid of a given jointless signed graph Σ.

Algorithm 2: BINARY

Input: A binary non-graphic matroid M given by an independence oracle and a jointless signed graph Σ such that $E(M) = E(\Sigma)$
Output: $M \cong M(\Sigma)$ or not
Determine a basis $\mathcal{B}$ of M
if $\mathcal{B}$ is not a spanning 1-tree in Σ **then**
  $M \ncong M(\Sigma)$
else
  if some $P_f \cup \{f\}$ where $f \in E(M) - \mathcal{B}$ with respect to $\mathcal{B}$ is not a positive cycle or a type I handcuff in Σ **then**
    $M \ncong M(\Sigma)$
  else
    if the incidence matrix A_Σ of Σ is not totally unimodular **then**
      $M \ncong M(\Sigma)$
    else
      $M \cong M(\Sigma)$
    end if
  end if
end if

As regards the proof of correctness of Binary Algorithm 2, every basis $\mathcal{B}$ of a binary signed-graphic and non-graphic matroid M corresponds to a spanning 1-tree in the signed graph representing it (Proposition 24). By Proposition 21, each fundamental circuit C_f of M with respect to $\mathcal{B}$ is either a positive cycle or a type I handcuff in the signed-graphic representation of M. Furthermore the incidence matrix of a signed graph representing M is totally unimodular by Corollary 6. Thereby the correctness of Binary Algorithm 2 results from Theorems 42 and 43.

Binary Algorithm 2 runs in polynomial time since we can check if $\mathcal{B}$ is a spanning 1-tree in Σ and if the set $P_f \cup \{f\}$ for each $f \in E(M) - \mathcal{B}$ is a positive cycle or a type I handcuff in Σ in polynomial time. Moreover, using Truemper's algorithm in [58] we can test in polynomial time if the incidence matrix of Σ is totally unimodular.

The following algorithm tests whether a binary matroid given by an independence oracle is signed-graphic.

Given a basis $\mathcal{B}$ of a binary matroid M, there is no known algorithm for testing the existence of a negative 1-tree T with $\mathcal{B} = E(T)$ such that the sets P_f are either path or a negative 1-path in T. Finding a polynomial time algorithm for

Algorithm 3: BINARY SIGNED-GRAPHIC

Input: A connected binary and non-graphic matroid M given by an independence oracle.

Output: M is signed-graphic or not.

Determine a basis $\mathcal{B}$ of M
if there is no negative 1-tree T such that $\mathcal{B} = E(T)$ **then**
  M is not signed-graphic
else
  if some P_f where $f \in E(M) - \mathcal{B}$ is neither a path nor a negative 1-path in T **then**
    M is not signed-graphic
  else
    Construct a signed graph Σ as follows: add to T each $f \in E(M) - \mathcal{B}$ as a link with end-vertices the ends of P_f (resp. $P_f - C_T$) if P_f is a path (resp. negative 1-path) and with the same sign with P_f
    if the incidence matrix A_Σ of Σ is not totally unimodular **then**
      M is not signed-graphic
    else
      M is signed-graphic
    end if
  end if
end if

determining the existence of such a negative 1-tree, will result in a polynomial time recognition algorithm for the class of binary signed-graphic matroids. The proof of correctness of Binary Signed-graphic Algorithm 3 is a straightforward consequence of Theorem 43.

Chapter 7

Quaternary signed-graphic matroids

We decompose the class of quaternary signed-graphic matroids extending Papalamprou and Pitsouli's decomposition theorem for binary signed-graphic matroids [40]. To this end, we use the operation of star composition which was defined in Chapter 4 and technical results which were obtained. Moreover, structural results concerning hereditary properties of cocircuits which were presented in Chapter 5, are of central importance for the decomposition. Non-graphic cocircuits of signed-graphic matroids whose signed-graphic representations are cylindrical or have a balancing vertex or are isomorphic to T_6 up to deleting joints are proved to be bridge-separable. Furthermore cocircuits with all-avoiding bridges of highly connected quaternary signed-graphic matroids are shown to correspond to the star of a vertex in the associate signed-graphic representation.

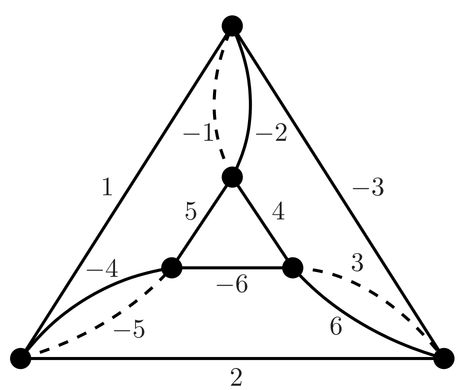


Figure 7.1: The signed graph T_6

7.1 Signed-graphic matroids with a minor isomorphic to $M(T_6)$

The structural properties of a signed-graphic matroid $M(\Sigma)$ such that $\Sigma\backslash J_\Sigma \cong T_6$ are examined in this section. Specifically, any non-graphic cocircuit of these matroids is proved to be bridge-separable. Additionally, if such a non-graphic cocircuit happens to have all-avoiding bridges then it is shown to correspond to the star of a vertex.

Lemma 7.1.1. *Let $M(\Sigma)$ be a signed-graphic matroid such that $\Sigma\backslash J_\Sigma \cong T_6$. If $Y \in \mathcal{C}^*(M(\Sigma))$ is non-graphic then Y is bridge-separable. Moreover, there exists a partition of the bridges of Y in $M(\Sigma)$ into two classes of all-avoiding bridges where one contains the separators of the balanced component of $\Sigma\backslash Y$ and the other the separators of the unbalanced component.*

Proof. Since Y is non-graphic, it is either an unbalancing bond or a double bond in Σ. By enumerating all possible cases, there is up to isomorphism one unbalancing bond e.g. $Y_1 = \{-3, 3, 6, 2\}$ in T_6 such that Y is non-graphic in $M(T_6)$ (see Figure 7.1). Thereby since $\Sigma\backslash J_\Sigma \cong T_6$, it can be easily deduced that the only unbalancing bonds in Σ are the stars of vertices without joints. If Y is an unbalancing bond in Σ, then $\Sigma\backslash Y$ is 2-connected and has two vertex-disjoint negative cycles. Thus, $\Sigma\backslash Y$ consists of one separator which is a non-binary bridge of Y in $M(\Sigma)$ and Y is trivially bridge-separable. If Y is a double bond in Σ, then up to isomorphism there is one double bond in T_6 e.g. $Y_2 = \{1, 2, 5, -6, -4\}$ such that Y is non-graphic in $M(T_6)$. Then either $\Sigma\backslash Y$ consists of two separators or Y is the star of a vertex containing joints in Σ. In both cases Y is trivially bridge-separable. $\square$

Lemma 7.1.2. *Let $M(\Sigma)$ be a signed-graphic matroid such that $\Sigma\backslash J_\Sigma \cong T_6$. If Y is a non-graphic cocircuit of $M(\Sigma)$ with all-avoiding bridges then Y is the star of a vertex in Σ.*

Proof. Suppose first that Y is an unbalancing bond in Σ. By enumerating all possible cases, there is up to isomorphism one unbalancing bond $Y_1 = \{-3, 3, 6, 2\}$ in T_6 such that Y is non-graphic in $M(T_6)$ (see Figure 7.1). Then $\Sigma\backslash Y$ has one 2-connected separator with two vertex-disjoint negative cycles which is a non-binary bridge of Y in $M(\Sigma)$. Thus, Y is the star of a vertex in Σ. Suppose now that Y is a double bond in Σ. Next it is shown that when Y is not the star of a vertex in Σ, then there are two separators in $\Sigma\backslash Y$ which correspond to overlapping bridges of Y in $M(\Sigma)$. A double bond of Σ contains only joints that are incident to vertices of the balanced component of $\Sigma\backslash Y$. Moreover, the number of these joints does not affect the avoidance of two bridges of Y in $M(\Sigma)$. Up to isomorphism there is one double bond in T_6 e.g. $Y_2 = \{1, 2, 5, -6, -4\}$, corresponding to a non-graphic cocircuit in $M(T_6)$. Then $\Sigma \backslash Y_2$ consists of two separators $B_1 = \{-5\}$, $B_2 = \{-1, -2, -3, 4, 6\}$. By the above, we shall consider only the case that $Y = Y_2 \cup J_1 \cup J_2$ where J_1, J_2 are sets of joints at the two vertices of B_1. Suppose that J_1 is the set of joints at the vertex v_1 of B_1 such that $star(v_1) = J_1 \cup \{5, -4, -5, -6\}$ and J_2 is the set of joints at the vertex v_2 such that $star(v_2) = J_2 \cup \{1, 2, -4, -5\}$. Then $\pi(M(\Sigma), B_1, Y) = \{\{1\}, \{5\}, \{-6\}, \{2\}, \{-4\}\}$ and $\pi(M(\Sigma), B_2, Y) = \{\{-4\}, \{5, -6\} \cup J_1, \{1, 2\} \cup J_2\}$, therefore B_1, B_2 overlap. $\square$

7.2 Bridge-separable cocircuits

Every cocircuit of a graphic matroid is bridge-separable as shown in [61].

Theorem 44 (Tutte [61]). *If Y is a cocircuit of a graphic matroid then Y is bridge-separable.*

Similarly, the bridge-separability property of non-graphic cocircuits in binary signed-graphic matroids is proved in [40].

Theorem 45 (Papalamprou, Pitsoulis [40]). *If Y is a non-graphic cocircuit of a binary signed-graphic matroid M, then Y is a bridge-separable cocircuit of M.*

However, Theorem 45 can be generalized to cover the whole class of quaternary signed-graphic matroids. As a first step, in Theorem 46, we prove such a result for signed-graphic matroids having a cylindrical signed graphical representation up to removing joints.

Theorem 46. *Let $M(\Sigma)$ be an internally 4-connected, quaternary and non-binary signed-graphic matroid and $\Sigma \backslash J_\Sigma$ be a cylindrical signed graph. If Y is a non-graphic cocircuit of $M(\Sigma)$ which corresponds to an unbalancing bond or a double bond whose balancing part has only joints, then Y is bridge-separable. Moreover, there exists a partition of the bridges of Y in $M(\Sigma)$ into two classes of all-avoiding bridges where one contains the separators of the balanced component of $\Sigma \backslash Y$ and the other the separators of the unbalanced components.*

Proof. We consider only the case where Y corresponds to a double bond whose balancing part has only joints, since an unbalancing bond is a double bond whose balancing set is empty of edges. By the definition of double bond, the signed graph $\Sigma \backslash Y$ consists of one balanced component and one or more unbalanced components.

Since Y is a non-graphic cocircuit, there is at least one unbalanced separator in $\Sigma\backslash Y$ corresponding to a non-graphic bridge of Y in $M(\Sigma)$, say B_0. Moreover, in $\Sigma\backslash Y$ let us denote by Σ^+ the balanced component and by Σ^- the unbalanced component containing B_0. Due to the fact that switching at vertices of Σ do not alter $M(\Sigma)$, we can assume that all the edges in the balanced separators of $\Sigma\backslash Y$ are positive. Let $\mathscr{U}^+$ be the class of the balanced separators of Σ^+ and $\mathscr{U}^-$ be the class of all the separators of the unbalanced components of $\Sigma\backslash Y$. Consider any pair of bridges B_1 and B_2 both belonging in either $\mathscr{U}^+$ or $\mathscr{U}^-$. Furthermore, let us denote by $v_1 \in V(B_1)$ and $v_2 \in V(B_2)$ the vertices of attachment such that B_2 is contained in $C(B_1, v_1)$ and B_1 is contained in $C(B_2, v_2)$, respectively. To prove the theorem it suffices to show that there exists an edge-set S_i in each $\Sigma.(B_i \cup Y)|Y$, $(i = 1, 2)$ which corresponds to a set in $\pi(M(\Sigma), B_i, Y)$ such that $S_1 \cup S_2 = Y$.

In what follows, we can assume that Σ is vertically 2-connected, since $M(\Sigma)$ is internally 4-connected. Moreover, since $\Sigma\backslash J_\Sigma$ is cylindrical we can assume that it is planar and, by Corollary 3, it can be assumed to have exactly two negative faces. We have the following cases:

Case 1: B_1 and B_2 are separators in $\mathscr{U}^+$

If B_1 and B_2 are separators of Σ^+, then they are both balanced. By the definition of contraction in signed graphs, since all the edges in the unbalanced components will be contracted, the signed graph $\Sigma.(B_1 \cup Y)|Y$ will consist only of half-edges attached at the vertices of attachment of B_1. The edges $Y(B_1, v_1)$ are half-edges attached at v_1 in that graph therefore, by Lemma 5.1.3, $S_1 = Y(B_1, v_1) \in \pi(M(\Sigma), B_1, Y)$. Similarly, we can find an edge-set $S_2 = Y(B_2, v_2) \in \pi(M(\Sigma), B_2, Y)$. We have that $V(\Sigma^+) \subseteq V(C(B_1, v_1)) \cup V(C(B_2, v_2))$, which implies that $Y(B_1, v_1) \cup Y(B_2, v_2) = Y$.

In what follows, B_1 and B_2 are separators of $\mathscr{U}^-$. Moreover, let $H_i \subseteq Y$ $(i = 1, 2)$ consist of all edges having an end-vertex to an unbalanced component of $\Sigma\backslash Y$ other than the one that contains B_i and the joints of the balancing part of Y.

Case 2: B_1 and B_2 are separators in different unbalanced components of $\Sigma\backslash Y$

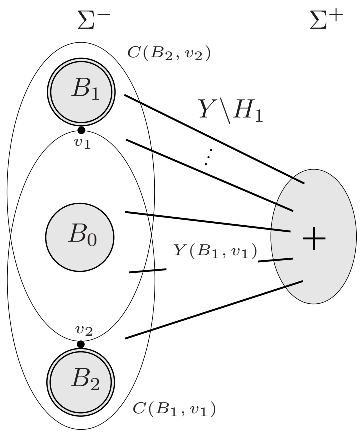
Due to the definition of contraction in signed graphs, the edges of H_i become half-edges with a common end-vertex in $\Sigma.(B_i \cup Y)|Y$. Then by Lemma 5.1.3, the edges of H_i are contained in some S_i of $\pi(M(\Sigma), B_i, Y)$ and it holds that $S_1 \cup S_2 = Y$.

Case 3: B_1 and B_2 are separators in the same unbalanced component of $\Sigma\backslash Y$

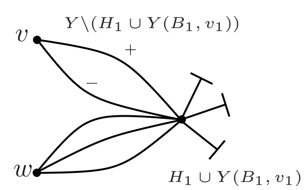
We can assume that B_1 and B_2 are separators of Σ^- since all other cases follow similarly. By Lemma 4.3.1, for every separator B of $\Sigma\backslash Y$ there exists at most one vertex of attachment v such that $Y(B, v)$ consists of links with different sign. Let by $v_1^{\pm}, v_2^{\pm}$ and $v_0^{\pm}$ denote these vertices of B_1, B_2 and B_0, respectively. We have the following cases:

Case 3.1: $B_1, B_2 \neq B_0$

The unbalanced separator B_0 may be contained in both $C(B_1, v_1)$ and $C(B_2, v_2)$ or in one of them. In the first case, all the edges of $\Sigma.(B_1 \cup Y)|Y$ share a common end-vertex and, moreover, the edges of H_1 and $Y(B_1, v_1)$ are half-edges (see Figure 7.2, where $v, w \neq v_1$). Therefore, by Lemma 5.1.3, there exists $S_1 \in \pi(M(\Sigma), B_1, Y)$ such that $S_1 = H_1 \cup Y(B_1, v_1)$. Similarly, we can find $S_2 \in \pi(M(\Sigma), B_2, Y)$ such that $S_2 = H_2 \cup Y(B_2, v_2)$. The set $Y(B_1, v_1) \cup Y(B_2, v_2)$ contains the edges of Y that have an end-vertex to Σ^- in Σ and thus, $S_1 \cup S_2 = Y$.



(a) signed graph Σ



(b) $\Sigma.(B_1 \cup Y)|Y$

Figure 7.2: Case 3.1 in proof of Theorem 46

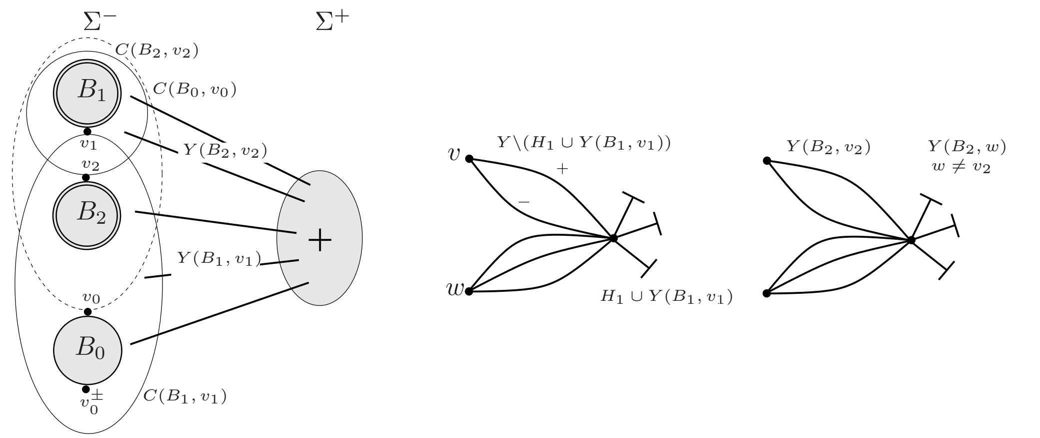
Consider now without loss of generality that B_0 is contained in $C(B_1, v_1)$ and not in $C(B_2, v_2)$. Then there must exist $v_0 \in V(B_0)$ such that B_1 and B_2 are contained in $C(B_0, v_0)$. We have the following subcases:

Case 3.1.a: $v_0 \neq v_0^{\pm}$

In this case either $C(B_1, v_1)$ or $C(B_2, v_2)$ is contained in $C(B_0, v_0)$; without loss of generality, consider the latter (see Figure 7.3(a)). Since $Y(B_2, v_2) \subseteq Y(B_0, v_0)$ the edges of $Y(B_2, v_2)$ have the same sign, thus, constitute a bond in $\Sigma.(B_2 \cup Y)|Y$ (see Figure 7.3(c)) implying that $Y(B_2, v_2) \in \mathcal{C}^*(M(\Sigma).(B_2 \cup Y)|Y)$. Then by Lemma 5.1.3, there exists $S_2 \in \pi(M(\Sigma), B_2, Y)$ such that $S_2 = Y(B_2, v_2)$. Given that B_0 is contained in $C(B_1, v_1)$, the edges in $Y(B_1, v_1)$ and the edges in H_1 are half-edges at a vertex in $\Sigma.(B_1 \cup Y)|Y$ (see Figure 7.3(b)). Therefore, by Lemma 5.1.3, there exists $S_1 \in \pi(M(\Sigma), B_1, Y)$ such that $S_1 = H_1 \cup Y(B_1, v_1)$. Since the edges of Y that have an end-vertex to Σ^- in Σ are contained in $Y(B_1, v_1) \cup Y(B_2, v_2)$, it follows that $S_1 \cup S_2 = Y$.

Case 3.1.b: $v_0 = v_0^{\pm}$

Assume that $v_2 \neq v_2^{\pm}$. Then $Y(B_2, v_2)$ consists of edges with equal signs, thus,



(a) signed graph Σ (b) $\Sigma.(B_1 \cup Y)|Y$ (c) $\Sigma.(B_2 \cup Y)|Y$

Figure 7.3: Case 3.1.a in proof of Theorem 46

by Lemma 5.1.3 there is $S_2 \in \pi(M(\Sigma), B_2, Y)$ such that $S_2 = Y(B_2, v_2)$. The edges in H_1 and the edges in $Y(B_1, v_1)$ are half-edges with common end-vertex in $\Sigma.(B_1 \cup Y)|Y$. Therefore, by Lemma 5.1.3, there is $S_1 \in \pi(M(\Sigma), B_1, Y)$ such that $S_1 = H_1 \cup Y(B_1, v_1)$.

Now assume that $v_2 = v_2^\pm$ which means that $Y(B_2, v_2^\pm)$ consists of edges with different sign (see Figure 7.4). By Lemma 4.3.1, the separator B_i can have only one vertex that is incident with edges of Y of different sign in $\Sigma.(B_i \cup Y)|Y$. Let this vertex be v_1' for B_1 and $Y^-(B_1, v_1')$ be the set of edges of the same sign, say negative, which are incident to v_1' in $\Sigma.(B_1 \cup Y)|Y$. Thus, $Y(B_2, v_2^\pm) = Y^+(B_2, v_2^\pm) \cup Y^-(B_1, v_1')$ where $Y^+(B_2, v_2^\pm)$ is the subset of the positive edges of $Y(B_2, v_2^\pm)$. The edges in H_1 and the edges in $Y(B_1, v_1)$ are half-edges with a common end-vertex in $\Sigma.(B_1 \cup Y)|Y$. Thereby, $H_1 \cup Y(B_1, v_1) \cup Y^-(B_1, v_1') = C_1^* \in \mathcal{C}^*(M(\Sigma).(B_1 \cup Y)|Y)$. Let v_2' be the vertex of attachment of B_2 such that $C(B_2, v_2')$ contains B_0; then, in $\Sigma.(B_2 \cup Y)|Y$, the edges in $Y(B_2, v_2')$ are half-edges. Moreover, the set $Y(B_2, v_2')$ is non-empty since Σ is vertically 2-connected. In $\Sigma.(B_2 \cup Y)|Y$, from the sets of parallel edges of Y only the set which has $v_2^\pm$ as a common end-vertex may consist of edges of different sign. Thus, $H_2 \cup Y^+(B_2, v_2^\pm) \cup Y(B_2, v_2') = C_2^* \in \mathcal{C}^*(M(\Sigma).(B_2 \cup Y)|Y)$. Since B_1, B_2 are balanced separators of Σ^-, $\Sigma.(B_i \cup Y)$ has as balancing vertex the common end-vertex of the edges of $Y(B_i, v_i)$, which implies that $M(\Sigma).(B_i \cup Y)$ is binary. Then $\pi(M(\Sigma), B_i, Y) = \mathcal{C}^*(M(\Sigma).(B_i \cup Y)|Y)$ and there are $S_1 = C_1^* \in \pi(M(\Sigma), B_1, Y)$ and $S_2 = C_2^* \in \pi(M(\Sigma), B_2, Y)$ such that $S_1 \cup S_2 = Y$.

Case 3.2: $B_2 = B_0$

We shall consider the following cases:

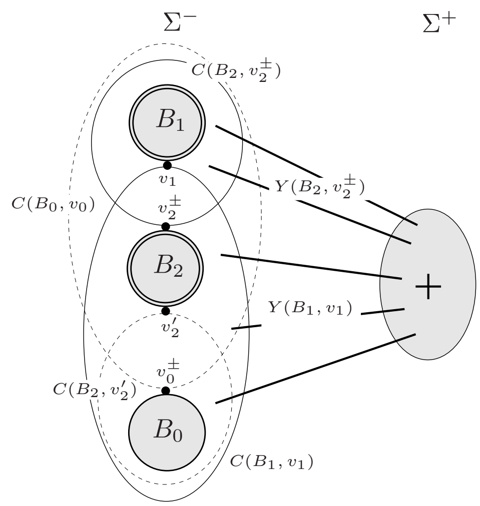


Figure 7.4: Case 3.1.b in proof of Theorem 46

Case 3.2.a: $v_2 \neq v_2^\pm$
In $\Sigma.(B_2 \cup Y)|Y$, the edges in $Y(B_2, v_2)$ are all those having the same sign and v_2 as an end-vertex; therefore, by Lemma 5.1.3, there is $S_2 \in \pi(M(\Sigma), B_2, Y)$ such that $S_2 = Y(B_2, v_2)$. In the signed graph $\Sigma.(B_1 \cup Y)|Y$, the edges in H_1 and the edges in $Y(B_1, v_1)$ are a set of half-edges attached at a common vertex. Therefore, by Lemma 5.1.3, there is $S_1 \in \pi(M(\Sigma), B_1, Y)$ such that $S_1 = H_1 \cup Y(B_1, v_1)$.
Case 3.2.b: $v_2 = v_2^\pm$
Assume first that B_1 has one balanced component $C(B_1, v_1^\pm)$ such that the edges in $Y(B_1, v_1^\pm)$ have different sign. Let $Y^+(B_i, v_i^\pm)$ and $Y^-(B_i, v_i^\pm)$ be the set of positive and negative edges of $Y(B_i, v_i^\pm)$, respectively, for $i = 1, 2$ (see Figure 7.5). Given that B_2 contains at least one negative face of Σ with no edges of Y, the unique negative face defined by the edges of $Y(B_1, v_1^\pm)$ has to be the outer face of Σ. This implies that all the edges of Y with end-vertex either in $V(C(B_1, v))$, where $v \neq v_1^\pm, v_1$, or in $V(C(B_1, v_1) \cap V(C(B_2, v_2^\pm))$ will have the same sign, say positive. This in turn implies that $Y^-(B_2, v_2^\pm) = Y^-(B_1, v_1^\pm)$. Examining the graphs $\Sigma.(B_i \cup Y)|Y$ we have that $H_1 \cup Y(B_1, v_1) \cup Y^-(B_2, v_2^\pm) = H_1 \cup Y(B_1, v_1) \cup Y^-(B_1, v_1^\pm) \in \mathcal{C}^*(M(\Sigma).(B_1 \cup Y)|Y)$ and $Y^+(B_2, v_2^\pm) \in \mathcal{C}^*(M(\Sigma).(B_2 \cup Y)|Y)$. Since B_1 is a balanced separator of Σ^-, the signed-graph $\Sigma.(B_1 \cup Y)$ has a balancing vertex, which implies that the corresponding signed-graphic matroid $M(\Sigma.(B_1 \cup Y))$ is binary. Thereby $\pi(M(\Sigma), B_i, Y) = \mathcal{C}^*(M(\Sigma).(B_i \cup Y)|Y)$ and by Lemma 5.1.3, there are $S_1 \in \pi(M(\Sigma), B_1, Y)$ such that $S_1 = H_1 \cup Y(B_1, v_1) \cup Y^-(B_2, v_2^\pm)$ and $S_2 \in \pi(M(\Sigma), B_2, Y)$ such that $S_2 = Y^+(B_2, v_2^\pm)$. Finally, if $Y(B_1, v)$ for every $v \neq v_1$ has edges of the same sign, the proof is similar.

$\square$

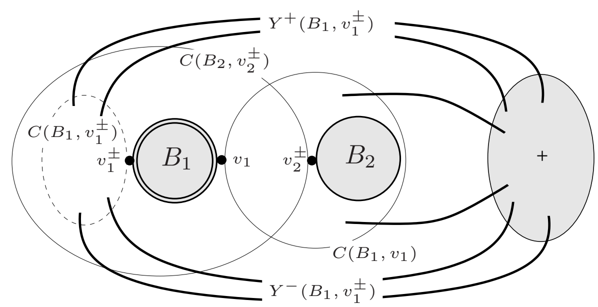


Figure 7.5: Case 3.2.b in proof of Theorem 46

We prove an analogous theorem for the case of quaternary signed-graphic matroids whose signed-graphic representations have a balancing vertex up to deletion of joints.

Theorem 47. *Let $M(\Sigma)$ be an internally 4-connected, quaternary and non-binary signed-graphic matroid and suppose that $\Sigma \backslash J_\Sigma$ has a balancing vertex. If Y is a non graphic cocircuit then Y is bridge-separable. Moreover, there exists a partition of the bridges of Y in $M(\Sigma)$ into two classes of all-avoiding bridges where one contains the separators of the balanced component of $\Sigma \backslash Y$ and the other the separators of the unbalanced components.*

Proof. Since Y is a non-graphic cocircuit of $M(\Sigma)$, it is either a double bond or an unbalancing bond in Σ. We consider only the case where Y corresponds to a double bond, since an unbalancing bond is a double bond with an empty balancing set. By the definition of double bond, the signed graph $\Sigma \backslash Y$ consists of one balanced component, denoted by Σ^+, and one or more unbalanced components. Furthermore, there is a non-graphic bridge B_0 of Y in $M(\Sigma)$ that corresponds to an unbalanced separator in $\Sigma \backslash Y$ which, by Proposition 12, contains a negative cycle other than joint. Due to the existence of B_0 and the fact that $\Sigma \backslash J_\Sigma$ has a balancing vertex, the balancing part of Y contains only joints. Therefore, there is only one unbalanced component of $\Sigma \backslash Y$ that is not joint unbalanced, denoted by Σ^- and, thus, B_0 corresponds to the unique unbalanced separator of Σ^-. Performing switchings at the vertices of Σ, all the edges of the balanced separators of $\Sigma \backslash Y$ become positive. Let $\mathscr{U}^+$ be the class of separators of Σ^+ and $\mathscr{U}^-$ be the class of separators of the unbalanced components of $\Sigma \backslash Y$. Also, let us consider any pair of bridges B_1 and B_2 both belonging to either $\mathscr{U}^+$ or $\mathscr{U}^-$. To prove the theorem it suffices to show that there exist $S_1 \in \pi(M(\Sigma), B_1, Y)$ and $S_2 \in \pi(M(\Sigma), B_2, Y)$ such that $S_1 \cup S_2 = Y$. In what follows, v_1 and v_2 are the vertices of attachment of

B_1 and B_2, respectively, such that $B_2 \subseteq C(B_1, v_1)$ and $B_1 \subseteq C(B_2, v_2)$. Moreover, Σ is vertically 2-connected, since $M(\Sigma)$ is internally 4-connected. We distinguish the following cases:

Case 1: B_1 and B_2 are separators of $\mathscr{U}^+$

Since B_1 and B_2 are separators of Σ^+, they are both balanced. By the definition of contraction in signed graphs, the signed graph $\Sigma.(B_1 \cup Y)|Y$ will consist of half-edges only attached at the vertices of attachment of B_1. Then the edges of $Y(B_1, v_1)$ are half-edges attached at v_1 in that graph and therefore, by Lemma 5.1.3, $S_1 = Y(B_1, v_1) \in \pi(M(\Sigma), B_1, Y)$. Similarly, we can find an edge-set $S_2 = Y(B_2, v_2) \in \pi(M(\Sigma), B_2, Y)$. We have that $V(\Sigma^+) \subseteq V(C(B_1, v_1)) \cup V(C(B_2, v_2))$, which implies that $Y(B_1, v_1) \cup Y(B_2, v_2) = Y$ and $S_1 \cup S_2 = Y$.

In what follows, let B_1 and B_2 be separators of $\mathscr{U}^-$.

Case 2: B_1 and B_2 are separators in different unbalanced components of $\Sigma \backslash Y$

Let H_i $(i = 1, 2)$ be the set of edges of Y in Σ that contains the edges of the balancing part of Y and the edges having an end-vertex to an unbalanced component of $\Sigma \backslash Y$ other than the one that contains B_i. In the signed graph $\Sigma.(B_i \cup Y)|Y$, the edges of H_i are half-edges with a common end-vertex. Then by Lemma 5.1.3 the set H_i is contained in an element S_i of $\pi(M(\Sigma), B_i, Y)$ and therefore, it holds that $S_1 \cup S_2 = Y$.

Case 3: B_1 and B_2 are separators in the same unbalanced component of $\Sigma \backslash Y$

Since $\Sigma \backslash J_\Sigma$ has a balancing vertex, an unbalanced component in $\Sigma \backslash Y$ either contains B_0 or it is joint unbalanced. We shall consider only the case where B_1 and B_2 are separators of Σ^- since the arguments for the other case are the same. We have the following two subcases:

Case 3.a: B_1 and B_2 are balanced separators of Σ^-

The unbalanced separator B_0 may be contained in both $C(B_1, v_1)$ and $C(B_2, v_2)$ or in one of them. In the first case, the edges of $Y(B_i, v_i)$ $(i = 1, 2)$, the edges of the balancing part of Y and the edges of Y that have an end-vertex to an unbalanced component of $\Sigma \backslash Y$ different from Σ^- become half-edges at a vertex in $\Sigma.(B_i \cup Y)|Y$. Then by Lemma 5.1.3, these half-edges are contained in some $S_i \in \pi(M(\Sigma), B_i, Y)$ and, therefore, $S_1 \cup S_2 = Y$. In the remaining case, assume that B_0 is contained in $C(B_1, v_1)$ and not in $C(B_2, v_2)$. Combining Lemma 4.3.2 and the fact that $\Sigma \backslash J_\Sigma$ has a balancing vertex, there is at most one vertex of attachment B_0, denoted by $v_0^{\pm}$, which is incident with edges of Y of different sign. Since $B_0 \nsubseteq C(B_2, v_2)$, the edges of $Y(B_2, v_2)$ have the same sign in $\Sigma.(B_2 \cup Y)|Y$ and therefore, there is $S_2 \in \pi(M(\Sigma), B_2, Y)$ such that $S_2 = Y(B_2, v_2)$. The edges of Y that have an end-vertex to an unbalanced component of $\Sigma \backslash Y$ different from Σ^-, the edges of the balancing part of Y and the edges of $Y(B_1, v_1)$ in Σ become half-edges incident at a vertex in $\Sigma.(B_1 \cup Y)|Y$. Hence by Lemma 5.1.3, there is $S_1 \in \pi(M(\Sigma), B_1, Y)$

that contains this set of half-edges and it holds that $S_1 \cup S_2 = Y$.
Case 3.b: $B_1 = B_0$ and B_2 is a balanced separator of Σ^-
Since $B_0 \subseteq C(B_2, v_2)$, the edges of $Y(B_2, v_2)$, the edges of Y that have an end-vertex to an unbalanced component of $\Sigma \backslash Y$ different from Σ^- and the edges of the balancing part of Y become half-edges incident at a vertex in $\Sigma.(B_2 \cup Y)|Y$. Then the set of these half-edges is contained in a set $S_2 \in \pi(M(\Sigma), B_2, Y)$. Now if we assume that $v_1 \neq v_0^{\pm}$ in Σ, then the edges of $Y(B_1, v_1)$ have the same sign in $\Sigma.(B_1 \cup Y)|Y$. Thus, there is $S_1 \in \pi(M(\Sigma), B_1, Y)$ such that $S_1 = Y(B_1, v_1)$ and $S_1 \cup S_2 = Y$. For the remaining case, i.e. when $v_1 = v_0^{\pm}$, the set $Y(B_1, v_1)$ contains edges of Y of different sign, i.e. $Y(B_1, v_1) = Y^+(B_1, v_1) \cup Y^-(B_1, v_1)$, where $Y^+(B_1, v_1)$ and $Y^-(B_1, v_1)$ are the sets of positive and negative edges of $Y(B_1, v_1)$, respectively. Then, by Lemma 5.1.3, there is $S_1 \in \pi(M(\Sigma), B_1, Y)$ such that $S_1 = Y^+(B_1, v_1)$. Since $Y(B_1, v_1)$ contains edges of Y of different sign, there are $y_1, y_2 \in Y$ such that $y_1 = \{v_1, w_1\}$ and $y_2 = \{v_2, w_2\}$ with positive and negative sign, respectively, where $w_i \in V(\Sigma^+$ and $v_i \in V(C(B_1, v_1))$. Moreover, $C(B_0, v_0^{\pm})$ is balanced and, therefore, y_1, y_2 are edges of a negative cycle in Σ. Since $\Sigma \backslash J_\Sigma$ has a balancing vertex, the edges of $Y^-(B_1, v_1) \subseteq Y(B_2, v_2)$ since otherwise, there are two vertex-disjoint negative cycles in $\Sigma \backslash J_\Sigma$. By Lemma 5.1.3, there is $S_2 \in \pi(M(\Sigma), B_2, Y)$ such that $Y^-(B_1, v_1) \subseteq S_2$ and, therefore, $S_1 \cup S_2 = Y$. □

7.3 Cocircuits with all-avoiding bridges

Regarding the class of graphic matroids, the property of all-avoiding bridges of a cocircuit is sufficient for a connected graphic matroid to have a graphical representation where the cocircuit is the star of a vertex as shown in [61].

Theorem 48 (Tutte, [61])**.** *Let Y be a cocircuit of a connected graphic matroid M such that any two bridges of Y avoid each other. Then there exists a 2-connected graph G where Y is the star of a vertex and $M = M(G)$.*

The above result was generalised for the class of binary signed-graphic matroids in [40].

Theorem 49 (Papalamprou, Pitsoulis [40])**.** *Let Y be a non-graphic cocircuit of a connected binary signed-graphic matroid M such that any two bridges of Y avoid each other. Then there exists a connected signed graph Σ where Y is the star of a vertex and $M = M(\Sigma)$.*

Let Y be a non-graphic cocircuit of an internally 4-connected quaternary signed-graphic matroid $M(\Sigma)$ with all-avoiding bridges; then, as shown in the following result, we can assume that Y is the star of a vertex in Σ.

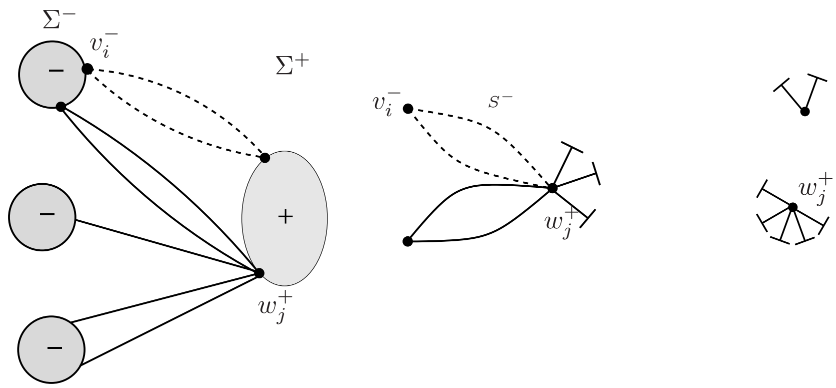
Theorem 50. *If Y is a non-graphic cocircuit with all-avoiding bridges of an internally 4-connected quaternary non-binary signed-graphic matroid $M(\Sigma)$, then Y is the star of a vertex in the signed graph Σ.*

Proof. The matroid $M(\Sigma)$ is internally 4-connected and by definition of biconnectivity, Σ is 3-biconnected. Due to the fact that Y is a non-graphic cocircuit of $M(\Sigma)$, it is a non-balancing bond in Σ. Moreover, the signed graph $\Sigma \backslash Y$ consists of one balanced component Σ^+ and one or more unbalanced components from which one contains an unbalanced separator denoted by B^-. Let Σ^- be the unbalanced component of $\Sigma \backslash Y$ that contains B^-. Performing switchings at the vertices of Σ, all the edges of the balanced separators of $\Sigma \backslash Y$ become positive. Thus only Y and the unbalanced separators of $\Sigma \backslash Y$ may have edges of negative sign. Assume on the contrary that Y is not the star of a vertex in Σ, equivalently $|E(\Sigma^+)| > 0$. Then there is a balanced separator B^+ that is a balanced block of Σ^+. Due to 3-biconnectivity of Σ, B^+ cannot have one vertex of attachment. Thus B^+ has two or more vertices w_j^+, $j = 1, \ldots, m$ such that $Y(B^+, w_j^+) \neq \emptyset$ (Figure 7.6 (a)). By hypothesis B^+, B^- are avoiding bridges of Y in $M(\Sigma)$, which implies that there are nonempty $S^+ \in \pi(M(\Sigma), B^+, Y)$ and $S^- \in \pi(M(\Sigma), B^-, Y)$ such that $S^+ \cup S^- = Y$. By Lemma 5.1.3, the elements of S^+ correspond either to half-edges incident at a vertex of attachment w_j^+ of B^+ or to a class of parallel links of the same sign incident at two vertices of attachment of B^+ in $\Sigma.(B^+ \cup Y)|Y$. We distinguish the following two cases:

<u>**Case 1:**</u> The elements of S^+ correspond to half-edges incident at a vertex of attachment w_j^+ in $\Sigma.(B^+ \cup Y)|Y$ (Figure 7.6 (c))

It holds that $S^+ = Y(B^+, w_j^+)$ for some vertex of attachment w_j^+ of B^+ in Σ (Figure 7.6 (a)). By Lemma 5.1.3 the elements of S^- correspond to either parallel links of the same sign incident at a vertex of attachment v_i^-, $i = 1, \ldots, n$ of B^- or half-edges incident at a vertex in $\Sigma.(B^- \cup Y)|Y$.

Let us assume first that the elements of S^- correspond to a parallel class of negative links incident to a vertex of attachment v_i^- in $\Sigma.(B^- \cup Y)|Y$ (Figure 7.6 (b)). The case where the elements of S^- correspond to a parallel class of positive links follows similarly. By avoidance of B^+ and B^-, the elements of $Y \backslash S^+$ are contained in S^- and therefore, they correspond to edges of the same sign and with an end-vertex at $C(B^-, v_i^-)$ in Σ. Thus $S^- = Y^-(B^-, v_i^-)$, where $Y^-(B^-, v_i^-)$ denotes the set of parallel negative links of Y with an end-vertex at $C(B^-, v_i^-)$ in Σ. If Y is an unbalancing bond in Σ, then $(A_1, \overline{A_1})$ is 2-biseparation of Σ, where $A_1 = \Sigma^+ \cup Y^-(B^-, v_i^-) \cup C(B^-, v_i^-)$ with $\Sigma[A_1], \Sigma[\overline{A_1}]$ being a balanced and an unbalanced signed graph, respectively and $V(A_1) \cap V(\overline{A_1}) = \{w_j^+, v_i^-\}$ (Figure 7.6 (a)). Otherwise Y is a double bond in Σ and there are joints or links



(a) signed graph Σ (b) $\Sigma.(B^- \cup Y)|Y$ (c) $\Sigma.(B^+ \cup Y)|Y$

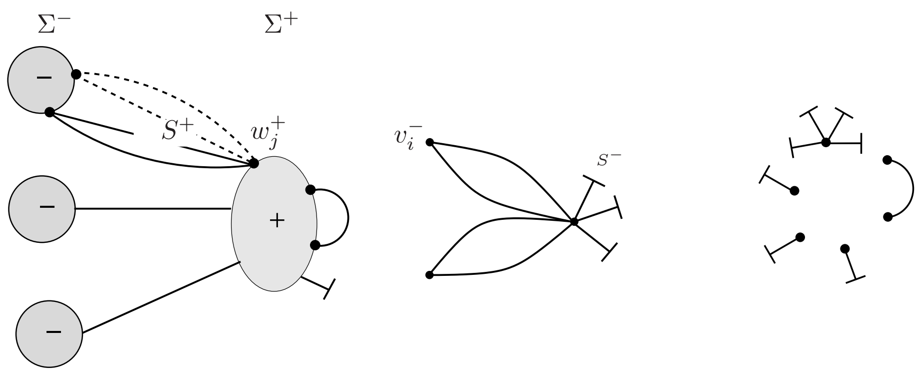
Figure 7.6: Case 1 in proof of Theorem 50

of the balancing part of Y that are contained to neither S^+ nor S^-, which is a contradiction.

Let assume now that the elements of S^- correspond to half-edges incident at a vertex in $\Sigma.(B^- \cup Y)|Y$ (Figure 7.7 (b)). The elements of S^- correspond to the edges of the balancing part of Y and to the links of the unbalancing part of Y having one end-vertex to some unbalanced component of $\Sigma \backslash Y$ different from the one that contains B^- in Σ. We note that $S^+ = Y(B^+, w_j^+)$ for some vertex of attachment w_j^+ of B^+ in Σ (Figure 7.7 (a)) and that Σ^- is the unbalanced component of $\Sigma \backslash Y$ that contains B^-. Then $(A_2, \overline{A_2})$ is a 2-biseparation of Σ, where $A_2 = \Sigma^- \cup Y(B^+, w_j^+) \cup C(B^+, w_j^+)$, since the signed graphs $\Sigma[A_2]$ and $\Sigma[\overline{A_2}]$ are both unbalanced and $V(A_2) \cap V(\overline{A_2}) = \{w_j^+\}$ (Figure 7.7 (a)). The above applies when Y is either a double bond or an unbalancing bond in Σ and it leads to a contradiction in both cases due to 3-biconnectivity of Σ.

<u>Case 2:</u> The edges of S^+ correspond to a class of parallel links of the same sign incident at two vertices of attachment of B^+ in $\Sigma.(B^+ \cup Y)|Y$

The elements of S^- correspond to either parallel links of the same sign incident at a vertex of attachment v_i^- of B^- or to half-edges incident at a vertex in $\Sigma.(B^- \cup Y)|Y$. Let us suppose first that the elements of S^- correspond to negative parallel links incident at a vertex of attachment v_i^- of B^- in $\Sigma.(B^- \cup Y)|Y$. By avoidance of B^+ and B^-, the elements of $Y \backslash S^+$ are contained in S^-. Moreover, they correspond to links of the same sign of the unbalancing part of Y with an end-vertex at $C(B^-, v_i^-)$ in Σ. Let Σ^- be the unbalanced component of $\Sigma \backslash Y$ that contains B^-. Then $(A_3, \overline{A_3})$ is 2-biseparation of Σ, where $A_3 = \Sigma^-$, since $\Sigma[A_3]$ and $\Sigma[\overline{A_3}]$ are both unbalanced signed graphs with $V(A_3) \cap V(\overline{A_3}) = \{v_i^-\}$. Furthermore Σ^- has at least two elements since it contains B^- that has a non-graphic minor. If the



(a) signed graph Σ (b) $\Sigma.(B^- \cup Y)|Y$ (c) $\Sigma.(B^+ \cup Y)|Y$

Figure 7.7: Case 1 in proof of Theorem 50

elements of S^- correspond to half-edges with a common end-vertex in $\Sigma.(B^- \cup Y)|Y$ then the links of the unbalancing part of Y that have an end-vertex to a connected component $C(B^-, v_i^-)$ belong to neither S^+ nor S^-, leading to a contradiction due to avoidance of B^+ and B^-. $\square$

Moreover, in the above proof, if we restrict Σ to be jointless, Y to be a non-balancing bond and replace the non-graphic bridge (B^-) with a graphic bridge then we obtain the following result.

Corollary 7. *If Σ is a jointless signed graph such that $M(\Sigma)$ is an internally 4-connected signed-graphic matroid and Y is a non-balancing bond in Σ and a cocircuit with all-avoiding bridges of $M(\Sigma)$, then Y is the star of a vertex in Σ.*

Finally, we prove a useful lemma for our decomposition approach regarding the existence of cocircuits with all-avoiding bridges in signed graphs after the deletion of joints.

Lemma 7.3.1. *Let Y be a cocircuit with all-avoiding bridges of a quaternary signed-graphic matroid $M(\Sigma)$ and a non-balancing bond in the signed graph Σ. If $Y' \subseteq Y \backslash J_\Sigma$ is a cocircuit of $M(\Sigma \backslash J_\Sigma)$, then Y' has all-avoiding bridges.*

Proof. Let B_1, B_2 be two avoiding bridges of Y in $M(\Sigma)$. By avoidance of B_1 and B_2, there are $S_1 \in \pi(M(\Sigma), B_1, Y)$ and $S_2 \in \pi(M(\Sigma), B_2, Y)$ such that $S_1 \cup S_2 = Y$. Let us assume first that $Y' = Y \backslash J_\Sigma$ which implies that none of B_1, B_2 corresponds to a joint unbalanced separator of $\Sigma \backslash Y$. Then by Lemma 5.1.5 (i), there are bridges B_1', B_2' of Y' in $M(\Sigma \backslash J_\Sigma)$ such that $B_i' \subseteq B_i \backslash J_\Sigma$. Moreover, there are $S_1' \in \pi(M(\Sigma \backslash J_\Sigma), B_1', Y')$ and $S_2' \in \pi(M(\Sigma \backslash J_\Sigma), B_2', Y')$ such that $S_1 \backslash J_\Sigma \subseteq S_1'$ and

$S_2 \backslash J_\Sigma \subseteq S_2'$. Thus since $S_1 \cup S_2 = Y$ and $Y' = Y \backslash J_\Sigma$, we have that $S_1' \cup S_2' = Y'$. Let us assume now that $Y' \subset Y \backslash J_\Sigma$ and that at least one of B_1 and B_2 corresponds to a joint unbalanced separator of $\Sigma \backslash Y$, let B_1. Then $B_1 \backslash J_\Sigma$ is incorporated to a balanced or an unbalanced separator B_1' of $\Sigma \backslash J_\Sigma \backslash Y'$ and since there is avoidance between any two separators of $\Sigma \backslash Y$, there is avoidance between any separator of $\Sigma \backslash J_\Sigma \backslash Y'$ and B_1'. $\square$

7.4 Structural results

Before the decomposition theorem of quaternary signed-graphic matroids which is presented in the next subsection, we provide known decomposition theorems and structural results which are used in its proof. Regarding the class of graphic matroids, the first decomposition theorem was provided by Tutte.

Theorem 51 (Tutte [61]). *Let M be a connected binary matroid and $Y \in \mathcal{C}^*(M)$ be a bridge-separable cocircuit, then M is graphic if and only if every Y-component of M is graphic.*

Tutte's decomposition theorem for graphic matroids was generalised by the decomposition theorem for binary signed-graphic matroids by Papalamprou and Pitsoulis.

Theorem 52 (Papalamprou, Pitsoulis [40]). *Let M be a connected binary matroid and $Y \in \mathcal{C}^*(M)$ be a bridge-separable cocircuit such that $M \backslash Y$ is not graphic. M is signed-graphic if and only if the Y-components of M are graphic except for one which is signed-graphic.*

Given a cocircuit Y of a matroid M, the connectivity of the Y-components is ensured by the following result.

Proposition 28 (Tutte [61]). *If Y is a cocircuit of a matroid M and B is a bridge of Y in M, then $M.(B \cup Y)$ is connected.*

We note that if $\mathscr{U}^+$ and $\mathscr{U}^-$ are two classes of bridges of a cocircuit Y of a matroid M, the sets U^+ and U^- denote the union of the bridges in the classes $\mathscr{U}^+$ and $\mathscr{U}^-$ respectively. The following two propositions generalize results from [61, 62].

Proposition 29. *If Y is a bridge-separable cocircuit of a matroid M with $\mathscr{U}^-$ and $\mathscr{U}^+$ the two classes of all-avoiding bridges, then $Y \in \mathcal{C}^*(M.(U \cup Y))$ where $U \in \{U^-, U^+\}$.*

Proof. The cocircuits of $M.(U\cup Y)$ are the circuits of $(M.(U\cup Y))^* = M^*|(U\cup Y)$, which are the cocircuits of M contained in $U\cup Y$, so Y is a cocircuit of $M.(U\cup Y)$. □

Proposition 30. *If M is a connected matroid and Y is a bridge-separable cocircuit of M with $\mathscr{U}^-$ and $\mathscr{U}^+$ the two classes of all-avoiding bridges, then $M.(U\cup Y)$ is connected where $U \in \{U^+, U^-\}$.*

Proof. Assume on the contrary that there is a separator $S \subseteq U^-\cup Y$ of $M.(U^-\cup Y)$. Then $r_{M.(U^-\cup Y)}(S) + r_{M.(U^-\cup Y)}((U^-\cup Y) - S) = r_{M.(U^-\cup Y)}(U^-\cup Y)$. Equivalently, $r_{M/U^+}(S) + r_{M/U^+}((U^-\cup Y) - S) = r_{M/U^+}(U^-\cup Y)$. Since $U^+ \subseteq E$ and $S \subseteq E - U^+$ and $(U^-\cup Y) - S \subseteq E - U^+$, by [[35] Proposition 3.1.6] it holds that $r_M(U^+\cup S) - r_M(U^+) + r_M(E-S) = r(E)$. Consider a basis B_S of $M|S$ and a basis B_{U^+} of $M|U^+$, then $r_M(S) = |B_S|$ and $r_M(U^+) = |B_{U^+}|$. Since U^+ and S are disjoint sets, $B_S\cup B_{U^+}$ is a basis of $M|(U^+\cup S)$ by definition of bases of matroids and therefore, $r_M(U^+\cup S) = |B_S\cup B_{U^+}|$. Furthermore $B_S\cap B_{U^+} = \emptyset$ and it follows that $|B_S\cup B_{U^+}| = |B_S| + |B_{U^+}| = r_M(S) + r_M(U^+)$. Thus, $r_M(S) + r_M(E-S) = r(E)$ which is a contradiction since M is connected. □

The property of non-graphicness is maintained for a cocircuit Y of some well-defined minor of a non-graphic matroid.

Lemma 7.4.1. *If Y is a non-graphic, bridge-separable cocircuit of M with $\mathscr{U}^-, \mathscr{U}^+$ the two classes of all-avoiding bridges of Y in M and $M.(U^+\cup Y)$ is graphic then Y is a non-graphic cocircuit of $M.(U^-\cup Y)$.*

Proof. By Proposition 29, Y is a cocircuit of $M.(U\cup Y)$ where $U \in \{U^-, U^+\}$. Since Y is a non-graphic cocircuit of M, the matroid $M\backslash Y$ contains a minor H isomorphic to one of the excluded minors of the class of graphic matroids, that is $F_7, F_7^*, M^*(K_5)$, $M^*(K_{3,3})$. By the fact that each of the above excluded minors is connected and H contains no element of Y, H is contained in a bridge of Y in M. Since Y is a graphic cocircuit in $M.(U^+\cup Y)$, we have that $M.(U^+\cup Y)\backslash Y = M/U^-\backslash Y = M\backslash Y/U^- = M\backslash Y.U^+ = M\backslash Y|U^+ = M|U^+$. Thereby $M|U^+$ is graphic which implies that H is not contained in any bridge of $\mathscr{U}^+$. Therefore H is a minor of a bridge of $\mathscr{U}^-$ and $M.(U^-\cup Y)\backslash Y = M|U^-$ is non-graphic. □

The bridges of a bridge-separable cocircuit Y of a matroid $M.(U\cup Y)$, where $\mathscr{U}$ is a class of all-avoiding bridges and U is the union of bridges in $\mathscr{U}$, are bridges of Y in the matroid M.

Proposition 31. *Let Y be a bridge-separable cocircuit of M with $\mathscr{U}^+, \mathscr{U}^-$ the two classes of all avoiding bridges. If B is a bridge of Y in M then B is a bridge of Y in $M.(U\cup Y)$ where $U \in \{U^+, U^-\}$.*

Proof. By Proposition 29, Y is a cocircuit of $M.(U \cup Y)$ where $U \in \{U^+, U^-\}$. It is enough to show that $M.(U \cup Y)\backslash Y)/B = (M.(U \cup Y)\backslash Y)\backslash B$. We have that $M.(U \cup Y)\backslash Y)/B = M.U/B = M.U/(U\backslash B) = M.(U\backslash B) = (M.U)\backslash B = (M.(U \cup Y)\backslash Y)\backslash B$. □

Proposition 32. *Let $M(\Sigma)$ be a connected quaternary signed-graphic matroid and Y be a double bond in the signed graph Σ whose balancing part contains links. If Σ^+ denotes the balanced component of $\Sigma\backslash Y$ then $M(\Sigma).(\Sigma^+ \cup Y)$ is either graphic or non-binary signed-graphic.*

Proof. Since $M(\Sigma)$ is connected, it follows that Σ is connected. Moreover, since Y is a double bond in Σ, $\Sigma\backslash Y$ consists of one balanced component and one or more unbalanced components. Perform switchings at the vertices of Σ so that all the edges of the balanced separators of $\Sigma\backslash Y$ become positive. Due to the definition of contraction in signed graphs and connectivity of Σ, the edges of the unbalancing part of Y become half-edges in $\Sigma.(\Sigma^+ \cup Y)$. Thus $\Sigma.(\Sigma^+ \cup Y)$ has at least one joint. By hypothesis and minimality of Y, the balancing part Y_W of Y contains links having both endvertices at Σ^+. Thus they are edges of negative cycles in the connected signed graph $\Sigma|(\Sigma^+ \cup Y_W)$. Since the class of signed-graphic matroids is closed under the operations of deletion and contraction, the matroid $M(\Sigma).(\Sigma^+ \cup Y)$ is signed-graphic. If $M(\Sigma).(\Sigma^+ \cup Y)$ is binary non-graphic then $\Sigma.(\Sigma^+ \cup Y)$ must be tangled, which is a contradiction since it contains a joint. Hence $M(\Sigma).(\Sigma^+ \cup Y)$ is either graphic or non-binary. □

7.5 Decomposition theorem

The main result of this work, i.e. a decomposition characterisation for the class of quaternary non-binary signed-graphic matroids, is provided in the theorem below. The decomposition is performed by deleting a non-graphic cocircuit and the blocks are shown to be signed-graphic of a particular structure.

Theorem 53. *Let M be an internally 4-connected quaternary non-binary matroid with not all-graphic cocircuits. Then M is signed-graphic if and only if*

(i) there is a non-graphic cocircuit Y of M which is bridge-separable with $\mathscr{U}^+, \mathscr{U}^-$ two classes of all-avoiding bridges where $\mathscr{U}^-$ contains all the non-graphic bridges,

(ii) $M.(\bigcup_{S \in \mathscr{U}^-} S \cup Y)$ is signed-graphic and $M.(\bigcup_{S \in \mathscr{U}^+} S \cup Y)$ is graphic.

Proof. Assume first that M is signed-graphic, then there is a connected signed graph Σ such that $M = M(\Sigma)$. Since M has not all-graphic cocircuits, it has a non-graphic cocircuit that corresponds to a non-balancing bond in Σ. If it corresponds to a double bond whose balancing part contains links, then by Lemma 5.1.1, there is another non-graphic cocircuit of $M(\Sigma)$ which corresponds to an unbalancing bond or a double bond whose balancing part contains only joints, denoted by Y. Thereby $\Sigma \backslash Y$ consists of a balanced component denoted by Σ^+ and one or more unbalanced components, where by Σ^- is denoted the subgraph that consists of all the unbalanced components. $M(\Sigma)$ is quaternary and non-binary and by Theorem 24, it follows that (1) $\Sigma \backslash J_\Sigma$ is cylindrical, or (2) $\Sigma \backslash J_\Sigma$ has a balancing vertex, or (3) $\Sigma \backslash J_\Sigma \cong T_6$ or (4) $\Sigma \backslash J_\Sigma = \Sigma_1 \oplus_k \Sigma_2$ for $k \in \{1, 2, 3\}$ where each $M(\Sigma_i), (i = 1, 2)$ is quaternary. Then for (1)-(3) by Theorem 46, 47 and Lemma 7.1.1, respectively, Y is bridge-separable and there is a partition of the bridges of Y in $M(\Sigma)$ into two classes $\mathscr{U}^+, \mathscr{U}^-$ of all-avoiding bridges where $\mathscr{U}^+$ contains the separators of Σ^+, while $\mathscr{U}^-$ contains the separators of Σ^-. Therefore $\mathscr{U}^-$ contains all the non-graphic bridges of $M(\Sigma) \backslash Y$. As regards case (4) then Σ has a minimal 3-biseparation (X_1, X_2) where $\Sigma[X_i]$ are vertically 2-connected and unbalanced and $|X_1| = 3$. Moreover, X_1 is the star of a vertex in Σ with one of its three elements being a joint, and therefore a cocircuit of $M(\Sigma)$. If X_1 is non-graphic, then $\Sigma[X_2]$ is not a B-necklace and therefore it is the unique unbalanced separator of $\Sigma \backslash X_1$. Thus X_1 is a non-graphic bridge-separable cocircuit of $M(\Sigma)$. Otherwise X_1 is graphic and by Proposition 12, $\Sigma[X_2]$ is either joint unbalanced or has a balancing vertex, since if it is balanced then $M(\Sigma)$ is graphic. Thereby $\Sigma \backslash J_\Sigma$ is a 2-vertex 2-sum of Σ_1 and Σ_2 where $M(\Sigma_i)$ are graphic. By hypothesis there is Y non-graphic cocircuit of $M(\Sigma)$ and by Proposition 17, there is $Y' \subseteq Y \backslash J_\Sigma$ cocircuit of $M(\Sigma \backslash J_\Sigma)$. We distinguish two cases either $Y' \subseteq E(X_2)$ or Y' is partitioned to $M(\Sigma_i)$. If $Y' \subseteq E(X_2)$, then Y' is a bridge-separable cocircuit in $M(\Sigma_2)$ by Lemma 5.2.5 and Theorem 44 and Y' is bridge-separable in $M(\Sigma \backslash J_\Sigma)$ by Theorem 34. Furthermore by Theorem 31, Y is bridge-separable in $M(\Sigma)$. Otherwise Y' is partitioned to $M(\Sigma_i)$ and $Y'_i \cup z$ where $Y' \cap X_i = Y'_i$ is a bridge-separable cocircuit of $M(\Sigma_i)$ by Lemma 5.2.5 and Theorem 44. By Theorem 34, Y' is a bridge-separable cocircuit of $M(\Sigma \backslash J_\Sigma)$ and by Theorem 31, Y is a bridge-separable cocircuit of $M(\Sigma)$. Furthermore by Theorem 17, the class of signed-graphic matroids is closed under the operation of deletion and contraction, therefore, the matroid $M(\Sigma).(U \cup Y)$ where $U \in \{U^-, U^+\}$ is signed-graphic and connected by Proposition 30. If Y is an unbalancing bond in Σ, then the signed graph $\Sigma.(\Sigma^+ \cup Y)$ is joint unbalanced and $M.(U^+ \cup Y) = M(\Sigma).(\Sigma^+ \cup Y)$ is graphic. The same holds when Y is a double bond whose balancing part contains only joints.

Conversely, assume that conditions (i) and (ii) hold. If there is one bridge B of

Y in M, then $M = M.(B \cup Y)$ is signed-graphic by hypothesis. Otherwise, there are at least two bridges and a non-graphic bridge in $\mathscr{U}^-$, since Y is a non-graphic cocircuit of M. The matroids $M.(U^- \cup Y)$ and $M.(U^+ \cup Y)$ are quaternary, as a minor of M and connected by Proposition 30. Moreover, $M.(U^- \cup Y)$ is signed-graphic and non-graphic having Y as a non-graphic cocircuit by Proposition 29 and the existence of a non-graphic bridge in $\mathscr{U}^-$. Therefore, there is a connected signed graph $\Sigma_1 = (G_1, \sigma_1)$ such that $M.(U^- \cup Y) = M(\Sigma_1)$. Then, by Theorem 24, one of the following holds: (1) $M(\Sigma_1)$ is binary or (2) $\Sigma_1 \backslash J_{\Sigma_1}$ is cylindrical, or (3) $\Sigma_1 \backslash J_{\Sigma_1}$ has a balancing vertex, or (4) $\Sigma_1 \backslash J_{\Sigma_1} \cong T_6$, or (5) $\Sigma_1 \backslash J_{\Sigma_1} = \Sigma'_1 \oplus_k \Sigma'_2$ for $k \in \{1, 2, 3\}$ where each $M(\Sigma'_i), i = 1, 2$ is quaternary. For (1) by Theorem 49, there is a signed-graphic representation of $M(\Sigma_1)$ where Y is the star of a vertex in Σ_1. For (2) and (3) by Theorem 50 and for (4) by Lemma 7.1.2, Y is the star of a vertex in Σ_1. Let us consider now case (5). Due to the fact that Y is a non-graphic cocircuit of $M(\Sigma_1)$ and Proposition 17, there is $Y' \subseteq Y \backslash J_{\Sigma_1}$ that is a non-balancing bond in $\Sigma_1 \backslash J_{\Sigma_1}$. Then Y' is partitioned to $M(\Sigma'_i)$ and for $k = 1$ by Lemma 5.2.2, for $k = 2$ by Lemma 5.2.5 and for $k = 3$ by Lemmas 5.2.12, 5.2.13, $Y_i \cup \overline{Z}$ is a cocircuit in $M(\Sigma'_i)$, where Z denotes the common elements of $M(\Sigma'_1)$ and $M(\Sigma'_2)$, $\overline{Z}$ contains every element z of Z such that $z \notin cl(X_i - Y_i)$, $E(M(\Sigma'_i)) = X_i \cup Z$ and $Y_i = E(M(\Sigma'_i)) \cap Y'$. Furthermore combining the fact that Y has all-avoiding bridges in $M(\Sigma_1)$ and Lemma 7.3.1, Y' has all-avoiding bridges in $M(\Sigma_1 \backslash J_{\Sigma_1})$. For $k = 1$ by Lemma 5.2.3, for $k = 2$ by Theorem 33 and for $k = 3$ by Theorem 36, $Y_i \cup \overline{Z}$ has all-avoiding bridges in the matroid $M(\Sigma'_i)$ that contains it. We can assume that each $\Sigma'_i \backslash J_{\Sigma'_i}$ is not a k-sum, since the decomposition of case (5) can be applied otherwise. Moreover, let us assume first that each $Y_i \cup \overline{Z}$ is a non-graphic cocircuit in $M(\Sigma'_i)$. Then one of the first four cases of Theorem 24 holds: for (1) by Theorem 50, for (2)-(3) by Theorem 49 and for (4) by Lemma 7.1.2, $Y_i \cup \overline{Z}$ is the star of a vertex in Σ'_i. Hence for $k = 1$ by Lemma 5.2.4 (i), for $k = 2$ by Lemmas 5.2.10 (i), 5.2.11 (i) and for $k = 3$ by Lemmas 5.2.18 (i), 5.2.19 (i), we have that Y' is the star of a vertex in $\Sigma_1 \backslash J_{\Sigma_1}$. Moreover, by Proposition 19, Y is the star of a vertex in Σ_1.

Let us assume now that each $Y_i \cup \overline{Z}$ is a graphic cocircuit in $M(\Sigma'_i)$, then there exists $(Y_i \cup \overline{Z})' \subseteq (Y_i \cup \overline{Z}) \backslash J_{\Sigma'_i}$ graphic cocircuit of $M(\Sigma'_i \backslash J_{\Sigma'_i})$ by Corollary 18. If each $(Y_i \cup \overline{Z})'$ is a balancing bond in $\Sigma'_i \backslash J_{\Sigma'_i}$, then by Proposition 20, $Y_i \cup \overline{Z}$ is a balancing bond in Σ'_i. Furthermore for $k = 1$ by Lemma 5.2.4 (ii), for $k = 2$ by Lemmas 5.2.10 (ii)-(iii), 5.2.11 (ii)-(iii), and for $k = 3$ by Lemmas 5.2.18 (ii)-(iii), 5.2.19 (ii)-(iii), Y' is a balancing bond in $\Sigma_1 \backslash J_{\Sigma_1}$ which implies that Y is a balancing bond in Σ_1, a contradiction. If each $(Y_i \cup \overline{Z})'$ is a non-balancing bond in $\Sigma'_i \backslash J_{\Sigma'_i}$, then by Lemma 7.3.1, it has all-avoiding bridges in $M(\Sigma'_i \backslash J_{\Sigma'_i})$. Furthermore by Corollary 7, each $(Y_i \cup \overline{Z})'$ is the star of a vertex in $\Sigma'_i \backslash J_{\Sigma'_i}$ and as above we have that Y is the star of a vertex in Σ_1. Otherwise $(Y_i \cup \overline{Z})'$ is a non-balancing bond in one of $\Sigma'_1 \backslash J_{\Sigma'_1}, \Sigma'_2 \backslash J_{\Sigma'_2}$ and a balancing bond in the other. Suppose that $(Y_1 \cup \overline{Z})'$ is a non-balancing bond in $\Sigma'_1 \backslash J_{\Sigma'_1}$ and $(Y_2 \cup \overline{Z})'$ is a balancing bond in $\Sigma'_2 \backslash J_{\Sigma'_2}$. Then by Corollary 7, $(Y_1 \cup \overline{Z})'$ is the star of a vertex in $\Sigma'_1 \backslash J_{\Sigma'_1}$ and it follows that $Y_1 \cup \overline{Z}$ is the star of a vertex in Σ'_1. On the other hand $(Y_2 \cup \overline{Z})'$ is a balancing bond in $\Sigma'_2 \backslash J_{\Sigma'_2}$ which implies that $Y_2 \cup \overline{Z}$ is a balancing bond in Σ'_2. For $k = 2$ by Lemma 5.2.11 (iv)-(v), and for $k = 3$ by Lemma 5.2.19 (iv)-(v), Y' is a balancing bond in $\Sigma_1 \backslash J_{\Sigma_1}$ which implies that Y is a balancing bond in Σ_1, a contradiction. The case where $Y_i \cup \overline{Z}$ is a non-graphic cocircuit in one of $M(\Sigma'_1), M(\Sigma'_2)$ and a graphic cocircuit in the other follows as the latter one. Regarding $M.(U^+ \cup Y)$, from (ii), we have that it is graphic and by Theorem 48, there is a connected signed graph $\Sigma_2 = (G_2, \sigma_2)$ such that $M.(U^+ \cup Y) = M(\Sigma_2)$, where Y is the star of a vertex in Σ_2.

Next we construct a signed graph $\Sigma = (G, \sigma)$ from the star composition of Σ_1 and Σ_2 with respect to a bond Y that is the star of a vertex in both Σ_1 and Σ_2. The underlying graph G is obtained from the graphs $G_1 \backslash Y$ and $G_2 \backslash Y$ as follows: (a) by adding a link between the end-vertex of the link of Y in G_1 and the end-vertex of the identical link of Y in G_2 or (b) by adding a joint at the end-vertex of the link of Y in G_2 when the identical element of Y in G_1 corresponds to a joint. The sign of an edge in Σ is the sign which is attributed to the edge by σ_1 when it belongs to G_1 and the sign which is attributed to an edge by σ_2 when the edge belongs to $G_2 \backslash Y$. Let us call Σ^+ the subgraph $\Sigma_2 \backslash Y$ upon the deletion of any isolated vertices and let us call Σ^- the subgraph of Σ that consists of the union of the unbalanced components of $\Sigma \backslash Y$, since Y is star in Σ_1. Hence Σ^+ is the balanced component of $\Sigma \backslash Y$ and Σ^- is the subgraph of Σ that consists of all the unbalanced components of $\Sigma \backslash Y$. Thus Y is a minimal set of edges in Σ, whose deletion increases the number of balanced components. Thereby Y is a non-balancing bond in Σ and a cocircuit of $M(\Sigma)$. By the above we derive the following equations

$$M(\Sigma).(U^- \cup Y) = M(\Sigma.(U^- \cup Y)) = M(\Sigma/U^+) = M(\Sigma_1) = M.(U^- \cup Y) \quad (7.1)$$

and

$$M(\Sigma).(U^+ \cup Y) = M(\Sigma.(U^+ \cup Y)) = M(\Sigma/U^-) = M(\Sigma_2) = M.(U^+ \cup Y). \quad (7.2)$$

By (7.2) we have that

$$M.(U^+ \cup Y) \backslash Y = M(\Sigma_2) \backslash Y = M(\Sigma_2 \backslash Y) = M(\Sigma^+). \quad (7.3)$$

Correspondingly by (7.1) it holds that $M.(U^- \cup Y) \backslash Y = M(\Sigma^-)$. Moreover, combining the fact that U^+ and U^- are the two separators of $M \backslash Y$ and the property that $M \backslash U^- = M/U^-$ we obtain that $M.(U^+ \cup Y) \backslash Y = M \backslash Y | U^+$. More precisely, $M.(U^+ \cup Y)\backslash Y= M/U^-\backslash Y= M\backslash Y\backslash U^-= M\backslash(U^- \cup Y)= M|U^+= M\backslash Y|U^+$. Next we show that $M(\Sigma)\backslash Y = M\backslash Y$. It holds that $M(\Sigma)\backslash Y= M(\Sigma\backslash Y)= M(\Sigma^+ \oplus_1 \Sigma^-)= M(\Sigma^+)\oplus_1 M(\Sigma^-)$ which is equal to $M.(U^+ \cup Y)\backslash Y \oplus_1 M.(U^- \cup Y)\backslash Y= M\backslash Y|U^+ \oplus_1 M\backslash Y|U^-= M\backslash Y$.

Then Y is a cocircuit in both $M(\Sigma), M$. Hence $E - Y$ is a hyperplane in both $M(\Sigma), M$. Therefore $r(M(\Sigma)\backslash Y)= r(M(\Sigma)) - 1$ and $r(M\backslash Y)= r(M) - 1$. By the fact that $M(\Sigma)\backslash Y = M\backslash Y$, we deduce that $r(M(\Sigma)\backslash Y)= r(M\backslash Y)$ and therefore $r(M(\Sigma)) = r(M)$. Next we shall show that $M(\Sigma) = M$ by showing that $\mathcal{I}(M(\Sigma)) = \mathcal{I}(M)$. Let $X \in \mathcal{I}(M(\Sigma))$ then $E - X$ is a cospanning set of $M(\Sigma)$ and spanning set in $M^*(\Sigma)$. Then $r(E - X) = r(M^*(\Sigma))$. Thus $r(E - X) = |E| - r(M(\Sigma))$. Since $r(M(\Sigma)) = r(M)$, $r(E - X) = |E| - r(M)$ and $r(E - X) = r(M^*)$. Therefore $E - X$ is a spanning set of M^* and a cospanning set of M, which in turn implies that $X \in \mathcal{I}(M)$. By reversing the arguments we show that $\mathcal{I}(M) \subseteq \mathcal{I}(M(\Sigma))$.

□

The following decomposition theorem for quaternary and non-binary signed-graphic matroids, which is based on Theorem 53, was stated by Dillon Mayhew. Its proof is very similar to the one presented above.

Theorem 54. *Let M contain a cocircuit, Y, such that deleting Y produces a non-graphic matroid. Then M is signed-graphic if and only if there exists such a cocircuit, Y, with the following properties: the connected components of $M\backslash Y$ can be partitioned into two classes, $\mathscr{U}^+, \mathscr{U}^-$, such that any two members of $\mathscr{U}^+$ or of $\mathscr{U}^-$ are avoiding, and whenever B is a bridge and $M.(B \cup Y)$ is non-graphic, then B is in $\mathscr{U}^-$. Furthermore, $M.(B \cup Y)$ is signed-graphic whenever B is in $\mathscr{U}^-$, and is graphic whenever B is in $\mathscr{U}^+$.*

In the following, we present an example of the decomposition of an internally 4-connected quaternary signed-graphic matroid $M(\Sigma)$.

Example 7.5.1. *Consider the signed-graphic matroid $M(\Sigma)$, where the signed graph Σ is depicted in Figure 7.9. The decomposition of the internally 4-connected quaternary signed-graphic matroid $M(\Sigma)$ is illustrated in Figure 7.8. The signed graph Σ is jointless and cylindrical. We note that cylidricality is maintained through k-sums. $M(\Sigma)$ has a non-graphic cocircuit $Y = \{3, -4, -1, -9, 5, 6\}$ which is bridge-separable and corresponds to an unbalancing bond in Σ. The bridges of Y in $M(\Sigma)$ are $B_1 = \{4\}$ and $B_2 = \{1, 2, -4, -5, -6, -2, -3, -8, 7\}$. The Y-components of $M(\Sigma)$ are the signed-graphic matroid $M(\Sigma).(B_2 \cup Y)$ and the graphic matroid $M(\Sigma).(B_1 \cup Y)$. The matroid $M(\Sigma).(B_1 \cup Y)$ is represented by the signed*

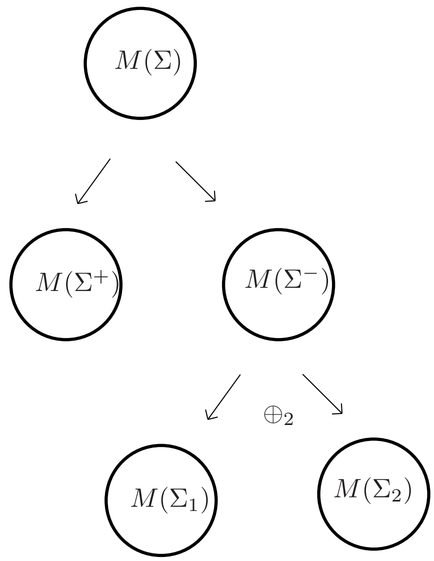


Figure 7.8: The decomposition of the signed-graphic matroid $M(\Sigma)$

graph Σ^+ which is depicted in Figure 7.10(a), while the matroid $M(\Sigma).(B_2 \cup Y)$ is represented by the signed graph Σ^- which is depicted in Figure 7.10(b).

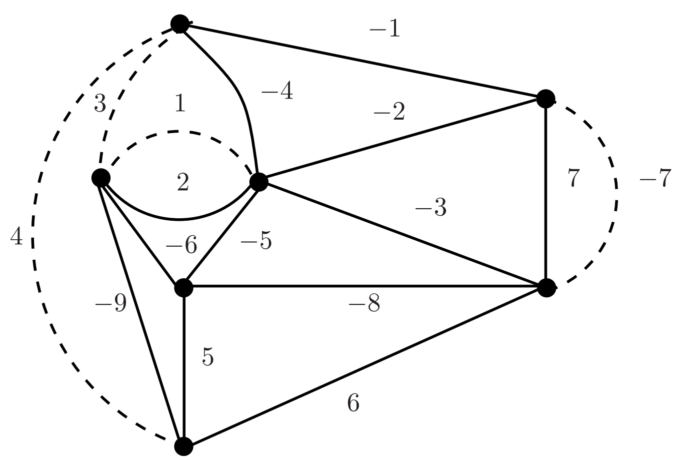
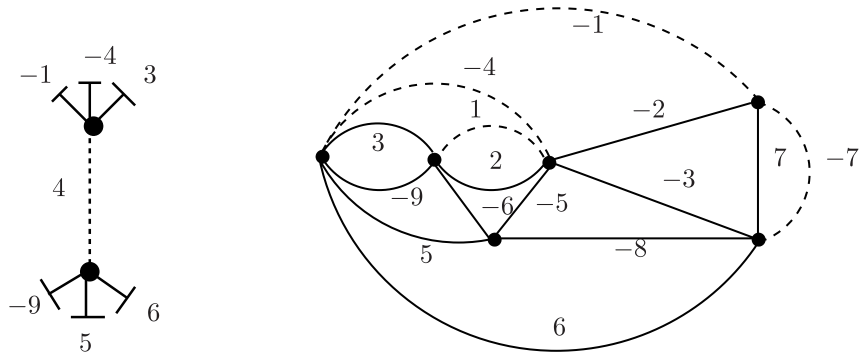


Figure 7.9: The signed graph Σ

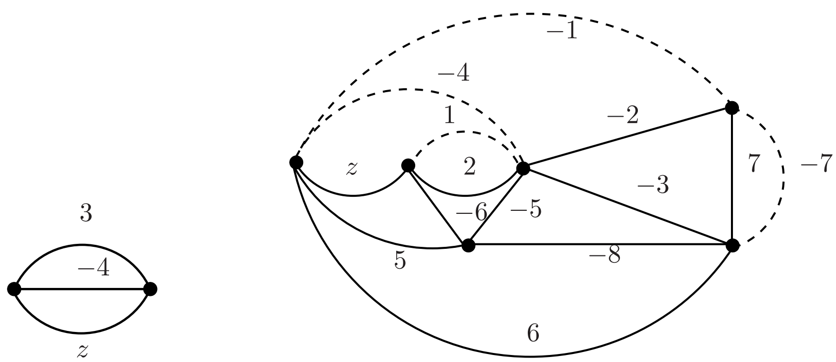
The signed-graphic matroid $M(\Sigma).(B_2 \cup Y) = M(\Sigma^-)$ is 2-connected, quaternary and non-binary and the signed graph Σ^- is 2-vertex 2-sum of Σ_1 and Σ_2 along z. The signed graph Σ_1 is depicted in Figure 7.11(a) and the signed graph Σ_2 is depicted in Figure 7.11(b). Furthermore Y is a cocircuit of $M(\Sigma^-)$ with all-avoiding bridges and a star bond in Σ^-. The signed-graphic matroid $M(\Sigma_2)$ is internally

4-connected quaternary and non-binary. Moreover, Y is a non-graphic cocircuit of $M(\Sigma_2)$ with all-avoidng bridges, therefore, by Theorem 50, Y is the star of a vertex.



(a) The signed graph Σ^+ (b) The signed graph Σ^-

Figure 7.10: Y-components of $M(\Sigma)$



(a) signed graph Σ_1 (b) signed graph Σ_2

Figure 7.11: The signed graphs Σ_1 and Σ_2

Chapter 8

Conclusions

In this chapter we point out the main results of the thesis and describe our contribution. Moreover, we draw conclusions from our research and suggest directions for future research.

The main target of our reseach was to characterize quaternary signed-graphic matroids by providing a decomposition theorem. To this aim we studied signed graphs which represent quaternary signed-graphic matroids obtaining structural results for the latter class of matroids. As a result, we established necessary and sufficient conditions for a quaternary matroid to be signed-graphic in Theorem 53 and we determined the building blocks of quaternary signed-graphic matroids. The decomposition theorem obtained generalizes Papalamprou and Pitsouli's decomposition theorem for binary signed-graphic matroids and constitutes the theoretical basis for a recognition algorithm for the assocated class of matroids.

An important consequence of the decomposition of quaternary signed-graphic matroids is that it will lead to the decomposition of larger classes of matroids, which have quaternary signed-graphic matroids as bulding blocks. Moreover, from such decomposition theorems, recognition algorithms for larger classes of matroids that contain quaternary signed-graphic matroids are expected to emerge. Since binet matrices are representation matrices for signed-graphic matroids over $\mathbb{R}$, another consequence of our decomposition theorem is that it will lead to a recognition algorithm for the subclass of binet matrices that represent quaternary signed-graphic matroids.

A characterization for cographic signed-graphic matroids with a nongraphic cocircuit is presented in Theorem 39. Specifically it is proved that a cographic matroid with a nongraphic cocircuit is signed-graphic if and only if each Fournier triple contains at most one nongraphic cocircuit. To achieve this, we proved that no cographic signed-graphic matroid contains a Fournier triple with two nongraphic cocircuits. Additionally, we showed that each cographic excluded minor of signed-graphic matroids contains a Fournier triple with two nongraphic cocircuits. Our characterization of cographic matroids apart from its structural and theoretical significance has also significant implications. It is expected to lead to a polynomial time recognition algorithm for the class of cographic and signed-graphic matroids and therefore to a polynomial time recognition algorithm for the class of binary signed-graphic matroids.

As concerns the structural properties of tangled signed graphs, we proved that the contraction of an m-edge is an operation which preserves the number of negative cycles. Moreover, we proved that the number of negative cycles in tangled signed graphs is polynomially bounded by the negative cycles of signed graphs in $\mathcal{P}(-K_4)$ and $\mathcal{P}(\pm C_3)$. As a consequence, condition (ii) of Theorem 10 in [39] can be checked in polynomial time. Thereby the polynomiality of General Recognition Algorithm of [39] is implied from the existence of a polynomial time recognition algorithm for binary signed-graphic matroids.

We provided a characterization for binary signed-graphic matroids which generalizes a well-known result for graphic matroids (Theorem 43). To this end we proved that the incidence matrices of tangled signed graphs are totally unimodular. What makes the difference between our characterization and existing characterizations for binary signed-graphic matroids is that it also provides a signed-graphic representation. From our characterization of binary signed-graphic matroids we derived two algorithms: a polynomial time algorithm for checking whether a binary and nongraphic matroid is isomorphic to the signed-graphic matroid of a given jointless signed graph (Binary Algorithm 2) and a recognition algorithm for binary signed-graphic matroids (Binary Signed-graphic Algorithm 3).

In the following, we suggest some directions for future research. The first suggestion concerns the recognition problem of binary signed-graphic matroids. Binary Signed-graphic Algorithm 3 has as input a binary matroid and decides whether the input matroid is signed-graphic. The main obstacle to polynomiality of the latter algorithm is step 1. Determining an algorithm, which given a basis $\mathcal{B}$ of a binary matroid, checks if there exists a negative 1-tree T with edgeset $\mathcal{B}$ such that the fundamental circuits with respect to $\mathcal{B}$ are paths or 1-paths in T, results in a polynomial time recognition algorithm for binary signed-graphic matroids. In addition it will imply a polynomial time recognition algorithm for determining whether a matroid is binary signed-graphic.

The characterization for quaternary signed-graphic matroids which is provided is about internally 4-connected matroids with not all-graphic cocircuits. Therefore it is of desire to obtain a recognition algorithm for quaternary signed-graphic matroids with all-graphic cocircuits or even an excluded minor characterization similar to that of the binary case. Another open question is the identification of the cocircuits which enable the decomposition of quaternary signed-graphic matroids or the decomposition of binary signed-graphic matroids. Although there exists a polynomial time method for finding a separating cocircuit, if there exists one, of a given binary matroid presented in [8] there is no polynomial time method that, given a binary matroid finds a separating and nongraphic cocircuit. Moreover, there is no polynomial time procedure that, given a quaternary matroid, tests the existence of a nongraphic and bridge-separable cocircuit.

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Appendix 1

Let Σ be a tangled signed graph that belongs to either $\mathcal{P}(-K_4)$ or $\mathcal{P}(\pm C_3)$. In this appendix, we prove that the number of negative cycles of Σ is polynomially bounded. The signed graphs $-K_4$ and $\pm C_3$ are depicted in Figure 8.1 and Figure 8.2, respectively.

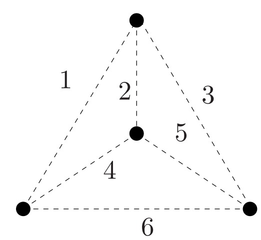

Figure 8.1: $-K_4$

Let us assume first the case where Σ belongs to $\mathcal{P}(-K_4)$. We denote by n_i the number of links of the parallel class of i, where $i = 1, \ldots, 6$. The number of positive links of n_i is denoted by n_i^+ while the number of negative links of n_i is denoted by n_i^-. The number of negative cycles of Σ of length j is denoted by $|C_{(j)}|$ $(j = 2, 3, 4)$.

- Negative cycles of Σ of length 2 (only parallel edges):

$|C_{(2)}| = \sum_{i=1}^{6} n_i^+ n_i^-$

- Negative cycles of Σ of length 3 $(\{1,2,5\}, \{2,3,4\}, \{3,1,6\}, \{4,5,6\})$:

$|C_{(3)}| = n_1^+(n_2^+ n_5^- + n_2^- n_5^+) + n_1^-(n_2^+ n_5^+ + n_2^- n_5^-)+$

$n_2^+(n_3^+ n_4^- + n_3^- n_4^+) + n_2^-(n_3^+ n_4^+ + n_3^- n_4^-)+$

$n_3^+(n_1^+ n_6^- + n_1^- n_6^+) + n_3^-(n_1^+ n_6^+ + n_1^- n_6^-)+$

$n_4^+(n_5^- n_6^+ + n_5^+ n_6^-) + n_4^-(n_5^+ n_6^+ + n_5^- n_6^-)$

- Negative cycles of Σ of length 4 ($\{1,2,4,6\},\{1,3,4,5\},\{2,3,6,5\}$):

$|C_{(4)}| = n_1^+(n_2^+(n_4^+ n_6^- + n_4^- n_6^+) + n_2^-(n_4^+ n_6^+ + n_4^- n_6^-))+$

$n_1^-(n_2^+(n_4^+ n_6^+ + n_4^- n_6^-) + n_2^-(n_4^+ n_6^- + n_4^- n_6^+))+$

$n_1^+(n_3^+(n_4^+ n_5^- + n_4^- n_5^+) + n_3^-(n_4^+ n_5^+ + n_4^- n_5^-))+$

$n_1^-(n_3^+(n_4^+ n_5^+ + n_4^- n_5^-) + n_3^-(n_4^+ n_5^- + n_4^- n_5^+))$

$n_2^+(n_3^+(n_5^+ n_6^- + n_5^- n_6^+) + n_3^-(n_5^+ n_6^+ + n_5^- n_6^-))+$

$n_2^-(n_3^+(n_5^+ n_6^+ + n_5^- n_6^-) + n_3^-(n_5^+ n_6^- + n_5^- n_6^-))$

Number of negative cycles of Σ= $|C_{(2)}| + |C_{(3)}| + |C_{(4)}|$

Let us assume now the case where Σ belongs to $\mathcal{P}(\pm C_3)$. We denote by n_k the number of links of the parallel class of k where $k = 1, \ldots, 3$. The number of positive links of n_k is denoted by n_k^+ while the number of negative links of n_k is denoted by n_k^-. The number of negative cycles of Σ of length l is denoted by $|C'_{(l)}|$ $(l = 2, 3)$.

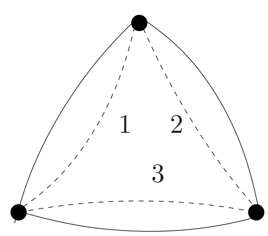

Figure 8.2: $\pm C_3$

- Cycles of Σ of length 2 (only parallel edges):

$|C'_{(2)}| = \sum_{i=1}^{3} n_i^+ n_i^-$

- Cycles of Σ of length 3:

$|C'_{(3)}| = n_1^+(n_2^+ n_3^- + n_2^- n_3^+)+$

$n_1^-(n_2^+ n_3^+ + n_2^- n_3^-)$

Number of negative cycles of $\Sigma = |C'_{(2)}| + |C'_{(3)}|$

Appendix 2

It is known that the matroids R_{15} and R_{16} are not cographic [**?**]. Moreover, among the cographic excluded minors of signed-graphic matroids, there are two, i.e., $M^*(G_{17})$ and $M^*(G_{19})$, that have all-graphic cocircuits. Let $\mathscr{M}$ be the class of cographic excluded minors of signed-graphic matroids with not all-graphic cocircuits, i.e., $\mathscr{M} = \{M^*(G_1), \ldots, M^*(G_{16}), M^*(G_{18}),\ M^*(G_{20}) \ldots, M^*(G_{29})\}$. For each cographic matroid $M \in \mathscr{M}$ a compact representation matrix over $GF(2)$ along with a Fournier triple, where two cocircuits are nongraphic, is provided in the following. The aforementioned Fournier triple was derived by checking the dual graphic matroid of each $M \in \mathscr{M}$ for a circuit C such that M/C has a minor isomorphic to $M(K_5)$ or $M(K_{3,3})$. Furthermore the following case analysis has been verified using the MACEK software [28].

The matroid $M^*(G_1)$

$$g_1^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{ccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \end{array}$$

(C^*, C_1^*, C_2^*) is a Fournier triple of $M^*(G_1)$ where $C^* = \{1, 4, -1\}$, $C_2^* = \{-1, -2, 2\}$ and $C_1^* = \{-1, -2, -3, -4, -5, -6, -7, 1\}$ are nonseparating cocircuits. Moreover, $M^*(G_1)\backslash C^*$ and $M^*(G_1)\backslash C_2^*$ have a $M^*(K_5)$ minor.

The matroid $M^*(G_2)$

$$g_2^* = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9 \end{array} \begin{array}{c} \begin{array}{cccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{array} \\ \begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&0&0&0&0&0\\ 0&0&1&1&1&1&1&1\\ 0&1&1&1&1&1&1&0\\ 0&0&0&0&1&1&1&0\\ 0&0&0&0&0&1&1&0\\ 0&0&0&1&1&1&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&0&0&1&1&0&0 \end{bmatrix} \end{array}$$

(C_1^*, C_8^*, C_9^*) is a Fournier triple of $M^*(G_2)$ where $C_1^* = \{-1,-2,-3,-4,-5,-6,-7,-8,1\}$, $C_8^* = \{-4,-5,8\}$ and $C_9^* = \{-5,-6,9\}$ are nonseparating cocircuits. Moreover, $M^*(G_2)\backslash C_8^*$ and $M^*(G_2)\backslash C_9^*$ have a $M^*(K_{3,3})$ minor.

The matroid $M^*(G_3)$

$$g_3^* = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array} \begin{array}{c} \begin{array}{cccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{array} \\ \begin{bmatrix} 0&1&1&1&1&1&1&1\\ 1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0\\ 0&0&1&1&1&1&1&0\\ 0&0&0&1&1&1&1&1\\ 0&0&0&1&1&0&0&0\\ 0&0&0&1&1&1&0&0\\ 0&0&0&0&1&1&1&0\\ 0&0&0&0&0&1&1&0\\ 0&0&0&0&1&1&0&0 \end{bmatrix} \end{array}$$

(C^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_3)$ where $C^* = \{-1,-2,3,4\}$, $C_2^* = \{-1,-2,-3,2\}$ and $C_3^* = \{-1,-2,-3,-4,-5,-6,-7,3\}$ are nonseparating cocircuits. Moreover, $M^*(G_3)\backslash C^*$ and $M^*(G_3)\backslash C_2^*$ have a $M^*(K_5)$ minor.

The matroid $M^*(G_4)$

$$g_4^* = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \end{array} \\ \left[\begin{array}{ccccccccc}
0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1\\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0\\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0\\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1\\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}\right] \end{array}$$

(C_6^*, C_7^*, C_8^*) is a Fournier triple of $M^*(G_4)$ where $C_6^* = \{-4,-5,-6,6\}$, $C_7^* = \{-4,-5,-7,7\}$ and $C_8^* = \{-5,-7,-8,8\}$ are nonseparating cocircuits. Moreover, $M^*(G_4)\backslash C_6^*$ and $M^*(G_4)\backslash C_7^*$ have a $M^*(K_{3,3})$ minor.

The matroid $M^*(G_5)$

$$g_5^* = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \begin{array}{c} \begin{array}{ccccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \end{array} \\ \left[\begin{array}{ccccccccc}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0\\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}\right] \end{array}$$

(C_3^*, C_6^*, C_8^*) is a Fournier triple of $M^*(G_5)$ where $C_3^* = \{-1,-2,-3,-4,-5,-6,-7,-8,3\}$, $C_6^* = \{-5,-6,-7,6\}$ and $C_8^* = \{-7,-8,-9,8\}$ are nonseparating cocircuits. Moreover, $M^*(G_5)\backslash C_6^*$ and $M^*(G_5)\backslash C_8^*$ have a $M^*(K_{3,3})$ minor.

The matroid $M^*(G_6)$

$$g_6^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(C_2^*, C_6^*, C_7^*) is a Fournier triple of $M^*(G_6)$ where $C_2^* = \{-1,-2,-3,-4,-5,-6,-7,-8,-9,2\}$, $C_6^* = \{-7,-8,-9,6\}$ and $C_7^* = \{-6,-7,-8,7\}$ are nonseparating cocircuits. Moreover, $M^*(G_6)\backslash C_6^*$ and $M^*(G_6)\backslash C_7^*$ have a $M^*(K_{3,3})$ minor.

The matroid $M^*(G_7)$

$$g_7^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccccc} -1 & -2 & -3 & -4 & -5 & -6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

(C_1^*, C_5^*, C_7^*) is a Fournier triple of $M^*(G_7)$ where $C_1^* = \{-1,-2,-3,-4,-5,1\}$, $C_5^* = \{-1,-2,5\}$ and $C_7^* = \{-2,-3,7\}$ are nonseparating cocircuits. Moreover, $M^*(G_7)\backslash C_5^*$ and $M^*(G_7)\backslash C_7^*$ have a $M^*(K_5)$ minor.

The matroid $M^*(G_8)$

$$g_8^* = \begin{array}{c|ccccccc} & -1 & -2 & -3 & -4 & -5 & -6 & -7 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 5 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 7 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 9 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 10 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}$$

(C^*, C_4^*, C_{10}^*) is a Fournier triple of $M^*(G_8)$ where $C_4^* = \{-1,-2,-3,-4,-5,-6,4\}$, $C^* = \{-4,5,6\}$ and $C_{10}^* = \{-4,-5,10\}$ are nonseparating cocircuits. Moreover, $M^*(G_8)\backslash C^*$ and $M^*(G_8)\backslash C_{10}^*$ have a $M^*(K_{3,3})$ minor.

The matroid $M^*(G_9)$

$$g_9^* = \begin{array}{c|cccccccc} & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \\ \hline 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 5 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 7 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 8 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}$$

(C^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_9)$ where $C^* = \{-1,-2,1,3\}$, $C_2^* = \{-1,-2,-3,2\}$ and $C_3^* = \{-1,-2,-3,-4,-5,-6,-7,3\}$ are nonseparating cocircuits. Moreover, $M^*(G_9)\backslash C^*$ and $M^*(G_9)\backslash C_2^*$ have a $M^*(K_{3,3})$ minor.

The matroid $M^*(G_{10})$

$$g_{10}^{*} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(C_1^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_{10})$ where $C_1^* = \{-1,-2,-3,-4,-5,-6,1\}$, $C_2^* = \{-1,-6,1,3\}$ and $C_3^* = \{-5,-6,1,2\}$ are nonseparating cocircuits. Moreover, $M^*(G_{10})\backslash C_2^*$ and $M^*(G_{10})\backslash C_3^*$ have a $M^*(K_5)$ minor.

The matroid $M^*(G_{11})$

$$g_{11}^{*} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(C_1^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_{11})$ where $C_1^* = \{-3,-6,5,7\}$, $C_2^* = \{-3,4,7\}$ and $C_3^* = \{-6,6,7\}$. Moreover, $M^*(G_{11})\backslash C_1^*$ and $M^*(G_{11})\backslash C_2^*$ have a $M^*(K_{3,3})$ minor.

The matroid $M^*(G_{12})$

$$g_{12}^* = \begin{array}{c c} & \begin{array}{cccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \end{array}$$

(C_1^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_{12})$ where $C_1^* = \{-1, -2, -3, -4, -5, -6, 1\}$, $C_2^* = \{-8, 1, 2, 3\}$ and $C_3^* = \{-1, 1, 3, 4\}$. Moreover, $M^*(G_{12})\backslash C_2^*$ and $M^*(G_{12})\backslash C_3^*$ have a $M^*(K_{3,3})$ minor.

The matroid $M^*(G_{13})$

$$g_{13}^* = \begin{array}{c c} & \begin{array}{cccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \end{array}$$

(C_1^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_{13})$ where $C_1^* = \{-1, -2, -3, -4, -5, -6, -7, 1\}$, $C_2^* = \{-6, 6, 7\}$ and $C_3^* = \{-6, 8, 9\}$ are nonseparating cocircuits. Moreover, $M^*(G_{13})\backslash C_2^*$ and $M^*(G_{13})\backslash C_3^*$ have a $M^*(K_{3,3})$ minor.

The matroid $M^*(G_{14})$

$$g_{14}^* = \begin{array}{c c} & \begin{array}{ccccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(C_1^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_{14})$ where $C_1^* = \{-7, -8, 2, 5\}$, $C_2^* = \{-7, -8, 4, -6\}$ and $C_3^* = \{2, 3, -7, -1\}$. Moreover, $M^*(G_{14})\backslash C_1^*$ and $M^*(G_{14})\backslash C_2^*$ have a $M^*(K_{3,3})$ minor.

The matroid $M^*(G_{15})$

$$g_{15}^* = \begin{array}{c c} & \begin{array}{ccccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(C_1^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_{15})$ where $C_1^* = \{-7, -8, 4, 2\}$, $C_2^* = \{-1, -2, -3, -4, -5, -6, -7, -8, 2\}$ and $C_3^* = \{-7, -8, -9, 3\}$ are nonseparating cocircuits. Moreover, $M^*(G_{15})\backslash C_1^*$ and $M^*(G_{15})\backslash C_3^*$ have a $M^*(K_{3,3})$ minor.

The matroid $M^*(G_{16})$

$$g_{16}^* = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \end{array} \\ \left[\begin{array}{ccccccccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

(C_1^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_{16})$ where $C_1^* = \{-8, 2, 3, 4\}$, $C_2^* = \{-4, -5, -6, -7, -8, 2\}$ and $C_3^* = \{1, -8, 4, -1\}$. Moreover, $M^*(G_{16})\backslash C_1^*$ and $M^*(G_{16})\backslash C_3^*$ have a $M^*(K_{3,3})$ minor.

The matroid $M^*(G_{18})$

$$g_{18}^* = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array} \begin{array}{c} \begin{array}{ccccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \end{array} \\ \left[\begin{array}{ccccccccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

(C_1^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_{16})$ where $C_1^* = \{-2, -3, -4, -5, -6, -7, -8, -9, 1\}$, $C_2^* = \{-4, -7, 4, 5\}$ and $C_3^* = \{-5, -6, -7, 5\}$ are nonseparating cocircuits. Moreover, $M^*(G_{18})\backslash C_2^*$ and $M^*(G_{18})\backslash C_3^*$ have a $M^*(K_{3,3})$ minor.

The matroid $M^*(G_{20})$

$$g_{20}^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(C^*, C_6^*, C_7^*) is a Fournier triple of $M^*(G_{20})$ where $C^* = \{-1, -2, 7, 8\}$, $C_6^* = \{-2, -3, -4, 6\}$ and $C_7^* = \{-1, -2, -3, -4, -5, -6, -7, 7\}$ are nonseparating cocircuits. Moreover, $M^*(G_{20})\backslash C^*$ and $M^*(G_{20})\backslash C_6^*$ have a $M^*(K_{3,3})$ minor.

The matroid $M^*(G_{21})$

$$g_{21}^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(C_1^*, C_2^*, C_5^*) is a Fournier triple of $M^*(G_{21})$ where $C^* = \{-4, 2, 5\}$, $C_2^* = \{3, 4, 5\}$ and $C_5^* = \{-1, -2, -3, 5\}$ are nonseparating cocircuits. Moreover, $M^*(G_{21})\backslash C^*$ and $M^*(G_{21})\backslash C_6^*$ have a $M^*(K_5)$ minor.

The matroid $M^*(G_{22})$

$$g_{22}^* = \begin{array}{c|c} & \begin{array}{cccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8 \end{array} & \begin{bmatrix} 1&1&1&0&0&0&0&0\\ 0&1&1&0&0&0&0&0\\ 1&1&0&0&0&0&0&0\\ 0&0&1&1&1&0&0&0\\ 0&1&1&1&1&1&0&0\\ 0&0&0&0&1&1&1&0\\ 0&0&0&0&0&1&1&1\\ 1&1&1&1&1&1&1&1 \end{bmatrix} \end{array}$$

(C_1^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_{22})$ where $C_1^* = \{-5,-6,-7,6\}$, $C_2^* = \{-6,-7,-8,7\}$ and $C_3^* = \{-1,-2,-3,-4,-5,-6,-7,-8,8\}$ are nonseparating cocircuits. Moreover, $M^*(G_{22})\backslash C_1^*$ and $M^*(G_{22})\backslash C_2^*$ have a $M^*(K_5)$ minor.

The matroid $M^*(G_{23})$

$$g_{23}^* = \begin{array}{c|c} & \begin{array}{ccccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6\\7 \end{array} & \begin{bmatrix} 1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&0&0\\ 0&1&1&1&1&1&1&1&0\\ 0&0&0&0&0&1&1&1&1\\ 0&0&1&1&1&0&0&0&0\\ 0&0&0&1&1&1&0&0&0 \end{bmatrix} \end{array}$$

(C_1^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_{23})$ where $C_1^* = \{-1,-8,3,4\}$, $C_2^* = \{-1,2,-9,4\}$ and $C_3^* = \{-1,-2,-3,-4,1\}$ are nonseparating cocircuits. Moreover, $M^*(G_{23})\backslash C_1^*$ and $M^*(G_{23})\backslash C_2^*$ have a $M^*(K_{3,3})$ minor.

The matroid $M^*(G_{24})$

$$g_{24}^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} -1 & -2 & -3 & -4 & -5 & -6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(C_1^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_{24})$ where $C_1^* = \{-1, 1, 9\}$, $C_2^* = \{-1, 4, -2\}$ and $C_3^* = \{-1, -2, -3, -4, -5, -6, 2\}$ are nonseparating cocircuits. Moreover, $M^*(G_{24})\backslash C_1^*$ and $M^*(G_{24})\backslash C_2^*$ have a $M^*(K_5)$ minor.

The matroid $M^*(G_{25})$

$$g_{25}^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{array} \begin{array}{c} \begin{array}{ccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(C_1^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_{25})$ where $C_1^* = \{-5, 10, 12\}$, $C_2^* = \{1, 3, 12\}$ and $C_3^* = \{-6, -7, 12\}$ are nonseparating cocircuits. Moreover, $M^*(G_{25})\backslash C_1^*$ and $M^*(G_{25})\backslash C_2^*$ have a $M^*(K_5)$ minor.

The matroid $M^*(G_{26})$

$$g_{26}^* = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11 \end{array} \begin{array}{c} \begin{array}{cccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{array} \\ \begin{bmatrix} 1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1&1\\ 1&1&1&0&0&0&0&0\\ 0&0&0&0&1&1&1&0\\ 0&1&1&1&1&1&0&0\\ 0&1&1&1&1&1&1&1\\ 0&0&1&1&1&0&0&0\\ 0&0&0&1&1&1&1&1\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&0&1&1&1 \end{bmatrix} \end{array}$$

(C_1^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_{26})$ where $C_1^* = \{-5, -6, -7, 4\}$, $C_2^* = \{-1, -7, 1, 5\}$ and $C_3^* = \{-1, -2, -3, -4, -5, -6, -7, 1\}$ are nonseparating cocircuits. Moreover, $M^*(G_{26})\backslash C_1^*$ and $M^*(G_{26})\backslash C_2^*$ have a $M^*(K_5)$ minor.

The matroid $M^*(G_{27})$

$$g_{27}^* = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array} \begin{array}{c} \begin{array}{cccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{array} \\ \begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&0&0&0&0\\ 0&1&1&1&1&1&0&0\\ 0&0&1&1&1&1&1&1\\ 0&0&1&1&1&1&1&0\\ 0&0&0&1&1&1&1&0\\ 0&0&0&1&1&1&1&1\\ 0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&1&1&1 \end{bmatrix} \end{array}$$

(C_1^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_{27})$ where $C_1^* = \{-1, -2, -3, -4, 2\}$, $C_2^* = \{-1, 2, -5, -6, 3\}$ and $C_3^* = \{-1, -2, -3, -4, -5, -6, -7, -8, 1\}$ are nonseparating cocircuits. Moreover, $M^*(G_{27})\backslash C_1^*$ and $M^*(G_{27})\backslash C_2^*$ have a $M^*(K_5)$ minor.

The matroid $M^*(G_{28})$

$$g_{28}^* = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 \end{array} \\ \left[\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0\\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right] \end{array}$$

(C_1^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_{26})$ where $C_1^* = \{-1,-2,-3,-4,-5,2\}$, $C_2^* = \{-1,2,-6,-7,3\}$ and $C_3^* = \{-1,-2,-3,-4,-5,-6,-7,-8,-9,1\}$ are nonseparating cocircuits. Moreover, $M^*(G_{28})\backslash C_1^*$ has a $M^*(K_5)$ minor and $M^*(G_{28})\backslash C_2^*$ have a $M^*(K_{3,3})$ minor.

The matroid $M^*(G_{29})$

$$g_{29}^* = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \begin{array}{c} \begin{array}{cccccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{array} \\ \left[\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0\\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1\\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right] \end{array}$$

(C_1^*, C_2^*, C_3^*) is a Fournier triple of $M^*(G_{26})$ where $C_1^* = \{-1,-2,-3,-4,-5,2\}$, $C_2^* = \{-2,-3,-9,-10,3,8\}$ and $C_3^* = \{-1,-2,1,4\}$ are nonseparating cocircuits. Moreover, $M^*(G_{28})\backslash C_1^*$ and $M^*(G_{28})\backslash C_2^*$ have a $M^*(K_{3,3})$ minor.

Appendix 3

In this appendix, we present representation matrices for some of the most known matroids that appear in the thesis.

The matroid $U_{2,4}$

$$u_{2,4} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

The Fano matroid F_7

$$f_7 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The dual of the Fano matroid F_7^*

$$f_7^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

The matroid $M^*(K_5)$

$$m_5^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

The matroid $M^*(K_{3,3})$

$$m_3^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The matroid R_{15}^*

$$r_{15}^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{array} \\ \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

The matroid R_{16}^*

$$r_{16}^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

The matroid T_6

$$t_6 = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

List of Symbols

A_Σ	incidence matrix of signed graph Σ	12
A_G	incidence matrix of graph G	7
$C(\mathcal{B}, f)$	fundamental circuit of f with respect to basis $\mathcal{B}$	19
$C(B, v)$	component determined by bridge B and vertex v	47
$C(T, f)$	fundamental cycle of f with respect to spanning tree T	19
G/X	contraction of $X \subseteq E(G)$ in graph G	8
$G.X$	contraction to $X \subseteq E(G)$ in graph G	8
$G \backslash X$	deletion of $X \subseteq E(G)$ in graph G	8
$G \vert X$	deletion to $X \subseteq E(G)$ in graph G	8
$G_1 \cong G_2$	G_1 isomorphic with G_2	7
$G_1 \cup G_2$	union of graphs G_1 and G_2	8
$GF(2)$	binary field	5
$GF(3)$	ternary field	5
$GF(4)$	quaternary field	5
I_n	$n \times n$ identity matrix	6
J_Σ	set of joints of a signed graph Σ	12
J_G	the set of half-edges and loops of graph G	7
M/X	the contraction of X from M	22
$M.X$	contraction to X in M	23
$M \backslash X$	the deletion of X from M	22
$M \vert X$	deletion to X in M	22
$M(G)$	cycle matroid of graph G	35
$M = (E, \mathscr{B})$	matroid M on E with bases family $\mathscr{B}$	19
$M = (E, \mathscr{C})$	matroid M on E with circuit family $\mathscr{C}$	18

$M = (E, \mathscr{I})$	matroid M on E with independence family $\mathscr{I}$	19
$M = (E, cl)$	matroid M on E with closure operator cl	20
$M = (E, r)$	matroid M on E with rank function r	20
$M[A]$	vector matroid of matrix A	28
M^*	dual matroid M	20
$M^*(G)$	bond matroid of graph G	36
$M_1 \cong M_2$	matroid M_1 isomorphic with M_2	21
$M_1 \oplus_1 M_2$	1-sum of matroids M_1 and M_2	26
$M_1 \oplus_2 M_2$	2-sum of matroids M_1 and M_2	26
$M_1 \oplus_3 M_2$	3-sum of matroids M_1 and M_2	27
N	network matrix	11
o	orientation of signed graph Σ	12
$si(M)$	simple matroid associated with matroid M	18
$U_{n,m}$	uniform matroid on m elements and rank n	19
$X \triangle Y$	symmetric difference of the sets X and Y	6

Glossary of terms

almost-regular matroids σχεδόν-κανονικά μητροειδή
avoidance αποφυγή
arc τόξο
balancing vertex κορυφή εξισορρόπησης
base βάση
binary matroid δυαδικό μητροειδές
bipartite διμερές
bond δεσμός
bridge γέφυρα
bridge B_1 avoids bridge B_2 η γέφυρα B_1 αποφεύγει τη γέφυρα B_2
bridge-separability διαχωρισμός γεφυρών
bridge-separable cocircuit συγκύκλωμα που διαχωρίζει τις γέφυρες
cardinality πληθικότητα
circuit κύκλωμα
closure κλειστότητα
closure operator τελεστής κλειστότητας
cobases συμβάσεις
cocircuit συγκύκλωμα
cographic συγγραφικά
coindependent συνανεξάρτητα
corank συμβαθμού
connected συνεκτικός
connected components συνεκτικές συνιστώσες
connectivity συνεκτικότητα
connectivity number αριθμός συνεκτικότητας
cocircuit αντικύκλωμα
coloop συμβρόχος
connected tree συνεκτικό δέντρο
compact representation matrix συμπιεσμένος πίνακας αναπαράστασης
contraction σύνθλιψη

cut περικοπή
cycle matroid μητροειδές κύκλου
cylindrical signed graphs κυλινδρικά προσημασμένα γραφήματα
decomposition theorem θεώρημα αποσύνθεσης
deletion of διαγραφή
deletion to περιορισμός
duality δυαδικότητα
dual δυικό
directed κατευθυνόμενος
elementary στοιχειώδης
excluded minor αποκλειόμενο έλασσον
faces όψεις
flat κλειστό (ή επίπεδο) σύνολο
Fournier triple Fournier τριάδα
$\mathbb{F}$-representable $\mathbb{F}$-αναπαριστώμενο
geometric dual γεωμετρικό δυικό
graphic matroids γραφικά μητροειδή
head κεφαλή
hyperplane υπερεπίπεδο
incidence matrix πίνακας πρόσπτωσης
joint άρθρωση
k-separation κ-διαχωρισμός
k-sums κ-αθροίσματα
k-bipartite κ-μερές
loop βρόχος
partition δυικότητα
pivoting οδήγηση
planar embedding επίπεδη αποτύπωση
matroid μητροειδές
Matroid Theory Θεωρία Μητροειδών
minor έλασσον
near-regular matroids περίπου-κανονικά μητροειδή
network matrix πίνακας δικτύου
non-separating cocircuit μη-διαχωριστικό συγκύκλωμα
quaternary matroid τετραδικό μητροειδές
rank βαθμός
rank function συνάρτηση βαθμού
regular matroids κανονικά μητροειδή
representability αναπαραστασιμότητα

signed-graphic matroid προσημασμένο-γραφικό μητροειδές
separator διαχωριστής
separating cocircuit διαχωριστικό συγκύκλωμα
span αστρική σύνθεση
standard representation matrix πρότυπος πίνακας αναπαράστασης
star composition αστρική σύνθεση
tangled signed graph περίπλοκο προσημασμένο γράφημα
totally unimodular matrix τ-πίνακας
coloop συμβρόχος
twisting περιστροφή
vector matroid μητροειδές διανύσματος
Y**-component** Υ-συνιστώσα

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