UNIVERSITY OF PATRAS

DOCTORAL THESIS

Open Quantum Systems and Applications to the Quantum Information Theory

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"When you ask what are electrons and protons I ought to answer that this question is not a profitable one to ask and does not really have a meaning. The important thing about electrons and protons is not what they are but how they behave, how they move. I can describe the situation by comparing it to the game of chess. In chess, we have various chessmen, kings, knights, pawns and so on. If you ask what chessman is, the answer would be that it is a piece of wood, or a piece of ivory, or perhaps just a sign written on paper, or anything whatever. It does not matter. Each chessman has a characteristic way of moving and this is all that matters about it. The whole game os chess follows from this way of moving the various chessmen."

Paul A.M. Dirac

Περίληψη

Η επιστήμη της κβαντικής πληροφορίας επιδιώκει την κατανόηση, στο ευρύτερο πλαίσιο της Κβαντομηχανικής, της πληροφορίας ως φυσικό αλλά και ως μαθηματικό εργαλείο. Έτσι, η κατανόηση των ιδιοτήτων της κβαντικής πληροφορίας (όπως π.χ. του εναγκαλισμού), κρίνεται απαραίτητη προϋπόθεση για την ανάπτυξη νέων κβαντικών τεχνολογιών. Στο πλαίσιο της διδακτορικής μου έρευνας ασχολήθηκα με α). την πλήρη κατανόηση και την περιγραφή της επικοινωνίας μεταξύ των απομακρυσμένων κβαντικών συστημάτων που αλληλεπιδρούν μέσω ενός κβαντικού πεδίου και β). την κατασκευή ενός θεωρητικού μοντέλου για την ακριβή περιγραφή του φαινομένου της μετάδοσης της πληροφορίας, η οποία δεν οδηγεί σε παραβίαση της αιτιότητας (Einstein causality). Για το σκοπό αυτό, στην παρούσα διατριβή μελετήθηκε το σύστημα των δύο εντοπισμένων ανιχνευτών (αρμονικών ταλαντωτών) που αλληλεπιδρούν μέσω ενός άμαζου βαθμωτού κβαντικού πεδίου, στην κατάσταση του κενού, μέσω μίας Unruh-DeWitt αλληλεπίδρασης. Το σύστημα αυτό είναι ισοδύναμο με ένα ανοικτό κβαντικό σύστημα (QBM model), όπου το πεδίο παίζει το ρόλο του περιβάλλοντος. Είναι ακριβώς επιλύσιμο και αποτελεί ένα μοντέλο κατάλληλο για την αντιμετώπιση θεμελιωδών προβλημάτων που αφορούν στις αλληλεπιδράσεις μεταξύ σωματιδίων και πεδίου, όπως το πρόβλημα της αιτιότητας (causality) και της τοπικότητας (locality) στις μετρήσεις κβαντικού πεδίου (quantum field measurements) που σχετίζονται και με τα πρόσφατα προτεινόμενα κβαντικά πειράματα στο διάστημα. Η ανάλυση της ακριβούς λύσης της χρονικής εξέλιξης του μοντέλου μας, οδήγησε στα ακόλουθα αποτελέσματα. i). Κοινές προσεγγίσεις που χρησιμοποιούνται για την μελέτη αντίστοιχων ανοικτών κβαντικών συστημάτων αποτυγχάνουν όταν η απόσταση μεταξύ των ανιχνευτών (συστημάτων) γίνεται ίση με την τάξη μεγέθους του χρόνου αποσύνθεσης (relaxation time) του συστήματος. Συγκεκριμένα, η μελέτη της δημιουργίας των συσχετισμών μεταξύ των απομακρυσμένων ανιχνευτών (συστημάτων) δεν περιγράφεται καλά από τη συνηθισμένη θεωρία διαταραχών (θεωρία διαταραχών 2ης τάξης) και την προσέγγιση Markov. ii). Υπάρχει μια μοναδική ασυμπτωτική κατάσταση στην οποία καταλήγει το υπό μελέτη σύστημα, η οποία είναι κατάσταση συσχετισμού (correlated state), όχι όμως κατάσταση εναγκαλισμού (entangled state), εκτός και αν η απόσταση μεταξύ των ανιχνευτών είναι τάξης μεγέθους του μήκους κύματος του ανταλλασσόμενου μεταξύ τους, κβάντου. iii). Τέλος, διαπιστώθηκε ότι η εξέλιξη των φαινομενικά εντοπισμένων παρατηρήσιμων μεγεθών

είναι μη-αιτιακή. Το τελευταίο είναι μια σημαντική επίδειξη του προβλήματος των δύο ατόμων του Fermi, σε ένα σύστημα που μπορεί να επιλυθεί με ακρίβεια. Υποστηρίζουμε ότι η έννοια του εναγκαλισμού στα σχετικιστικά συστήματα, και ειδικότερα η μελέτη της φυσικής σημασίας της εξαγωγής του εναγκαλισμού από το κενό (Harvesting) απαιτεί επανακαθορισμό λόγω του προβλήματος της αιτιότητας. Το αποτέλεσμα της έρευνας αυτής, αναμένεται να συμβάλλει στην ανάπτυξη του τομέα της κβαντικής πληροφορίας, μέσα από τα αποτελέσματα που αφορούν στην κατανόηση της κβαντικής επικοινωνίας σε μεγάλες αποστάσεις.

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Abstract

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Open Quantum Systems and Applications to the Quantum Information Theory

by THEODORA KOLIONI

This thesis is a contribution to the debate on (a) fully understanding and describing the communication between remote systems through quantum fields, and (b) constructing a theoretical model for accurately describing the information transmission, which does not lead to a violation of causality. To this end, we studied the system of two localized detectors (oscillators) interacting through a mass-less scalar quantum field in a vacuum state via an Unruh-DeWitt coupling. This system admits an exact solution is providing a good model for addressing fundamental issues in particle-field interactions, causality, and locality in quantum field measurements that are relevant to proposed quantum experiments in space. Our analysis of the exact solution led to the following results. (i) Common approximations used in the study of analogous open quantum systems fail when the distance between the detectors becomes of the order of the relaxation time. In particular, the creation of correlations between remote detectors is not well described by ordinary perturbation theory and the Markov approximation. (ii) There is a unique asymptotic state that is correlated; it is not entangled unless the detector separation is of the order of magnitude of the wavelength of the exchanged quanta. (iii) The evolution of seemingly localized observables is non-causal. The latter is a manifestation of Fermi's two-atom problem, albeit in an exactly solvable system. We argue that the problem of causality requires a re-examination of the notion of entanglement in relativistic systems, in particular, the physical relevance of its extraction from the quantum vacuum.

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List of Abbreviations

QBM	**Q**uantum **B**rownian **M**otion
OQS	**O**pen **Q**uantum **S**ystem
QME	**Q**uantum **M**aster **E**quation
TC2	second-order **T**ime **C**onvolution
PT	**P**artial **T**ranspose
CPTP	**C**ompletely **P**ositive **T**race **P**reserving
PPT	**P**ositive **P**artial **T**ranspose
WWA	**W**igner **W**eisskopf **A**pproximation
UR	**U**ncertainty **R**elation
QFT	**Q**uantum **F**ield **T**heory

Physical Constants

We work in the case where $\hbar = c = \omega = 1$.

List of Symbols

r	distance	m
t	time	sec
ω	angular frequency	rad
Ei	Exponential function	
Si	Sine Integral	
Ci	Cosine Integral	
Shi	Sinh Integral	
Tr	Trace	

Dedicated to Michail

Chapter 1

Introduction

Understanding how spatially separated quantum systems interact via relativistic quantum fields becomes increasingly important. Many proposed quantum experiments in space lie in the regime where relativistic effects become important. Our ability to construct entangled states of atoms at large separation will reach a regime where retarded propagation of photons will be a significant factors, thus, allowing us to explore experimentally the interplay between entanglement and relativistic causality. Furthermore, the interplay between localization and causality is a source of long-standing problem in the foundations of Quantum Field Theory (QFT).

In this research, we study an *exactly solvable* model that allows us to address issues such as the above. The model consists of two harmonic oscillators interacting with a quantum field through and Unruh-DeWitt coupling [1, 2, 3]. The field lies initially at the vacuum. The harmonic oscillators can be viewed as detectors or as crude approximations to atoms (N-level systems). Finding and analysing the exact solution to the system, we obtain the following results.

> [i]Usual approximations employed in the treatment of similar quantum systems (Markov approximation, perturbative master equation, Wigner-Weisskopf approximation) fail if the separation of the two detectors becomes of the order of relaxation time or larger. In particular, the above approximations break down completely in all processes that involve the exchange of information between the two subsystems. While this result is derived in a specific model system, the context in which it is obtained is quite generic for open quantum systems, and for this reason we believe that it has a broader applicability. It has important implications. For example, it suggests that at least some entangled states for atoms at large separations manifest significant deviations from exponential decay. There is a unique asymptotic state of the system that is correlated. Entanglement generation persists at times of the order of the relaxation scale, and hence, this result goes beyond most studies of entanglement generation (or harvesting) from the vacuum that rely on time-dependent perturbation theory. If we assume that the variables pertaining to detectors are localized quantum observables, then the reduced dynamics of the detector are non-causal. This is a common issue in analogous systems, like for example, the famous Fermi two-atom problem—see, below. Having an exact solution allows us to show that this behavior is not an artefact of an approximation in

the derivation of the dynamics. Rather, its origins are kinematical: we need to identify new observables that also involve the field degrees of freedom in order to describe localized measurements. This conclusion implies that entanglement generated between the detectors may not be a physically meaningful quantum resource to harvest.

The broader context of our results is the following. *Non-Markovian dynamics.*

A localized quantum system, such as an atom, in an excited state decays to the vacuum through its interaction with a quantum field, even if the latter is in the vacuum state. Such decays are typically exponential. When the system is treated using the theory of open quantum systems, the exponential decay law arises as a consequence of Markovian open system dynamics.

Markovian dynamics are generic for weak coupling of the system to environment. The second-order Markovian master equation become exact at the van Hove limit [4], where the system-environment coupling λ while the rescaled time $\lambda^2 t$ is constant [5]. It provides an excellent approximation for a large class of systems, especially in atom optics. However, comparison with exact solution of the evolution equations—as, for example, in quantum Brownian motion [6]—shows many regimes in which the second order master equation fails. In particular, the van Hove limit may not be physical relevant in cases where the open system dynamics are characterized by long time scales other than the dissipation time. This occurs for example, if the environment is characterized by resonance frequencies or thresholds [7]. In this paper, we present another case of failure of the Markovian approximate, due to the time-scale of traversal time in a bipartite system with components at large separation.

There is an increased emphasis in recent years towards understanding non Markovian dynamics in open quantum systems, because of their relevance to many physical contexts, for example, condensed matter physics, quantum control, quantum biology and quantum optics—see, [8] and references therein. Furthermore, our ability to prepare entangled state in multipartite systems provides novel technical and conceptual challenges to the theory of open quantum systems, because they go beyond the traditional paradigm of a central, localized system weakly interacting with an environment.

Consider, for example, two atoms prepared in an entangled state and separated by distance r and interacting with the quantum electromagnetic field. For small separations, this system is well described by the second order master equation, see, for example, Ref. [9]. However, as the separation increases, approximations involved in the derivations of the second order master equation break down, for example, the Rotating Wave Approximation [10] ,[11]. When r becomes comparable to the decay time of the atoms Γ^{-1}, the van Hove limit—descaling the time t but not the distance r—is not a useful approximation. Simply by analyzing the mathematical assumptions involved in the Markov approximation, we expect the decay of an entangled pair of atoms to be strongly non-Markovian, when Γr becomes of order unity or larger, or equivalently when the decay time scale is of the same order with

the retarded propagation time-scale. This expectation is verified by our analysis.

Note that this breakdown of Markovian behavior is a non-perturbative effect: Γ is proportional to the coupling constant, but we can always find a distance r such that $\Gamma r \simeq 1$. For quantum states relevant $\Gamma r \sim 1$ for r of the order of hundreds of meters or kilometers.

Fermi's two-atom system. Furthermore, the two-atom system is a classic example for understanding transmission of information through quantum fields that originates from Fermi [12]. He showed that the transmission of information between the two atoms occurs in accordance with Einstein locality, i.e., there is Fermi assumed that at time $t = 0$, atom A is in an excited state and atom B in the ground state. He asked when B will notice A and move from its ground state—see fig. (1.1). In accordance with Einstein locality, he found that this happens only at time greater than r. It took about thirty years for Shirokov to point out that Fermi's result is an artifact of an approximation [13], [14].

Several studies followed with conclusions depending on the approximations used. It was believed that non-causality is due to the use of bare initial states, and that it would not be present in a renormalised theory. However, Hegerfeld showed that non-causality is generic [15], [16], as it depends only on the assumption of energy positivity and on the existence of systems that are localized in disjoint spacetime regions—see also [17].

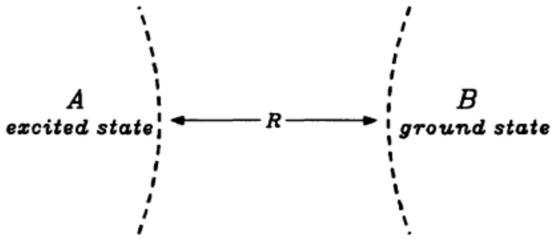


FIGURE 1.1: Two systems A and B in the distance r between them. For $t = 0$, the system A is located at the excited state and the system B at the ground state and initially do not interact with each other. [15]

Entanglement generation. It is well known that two systems that do not directly interact may become entangled through their interaction with a third system. This general result also applies to localised systems (detectors) interacting with the quantum field. The detectors may develop entanglement even if the field lies on its ground state [18]. This process is called *entanglement harvesting* and it has been extensively studied for different initial detector states, detector trajectories, or spacetime geometries—see, for example, [19, 20, 21]. Interestingly, this process of entanglement creation may also take

place between objects that remain spacelike separated, i.e., in some models, entanglement is seemingly generated outside the lightcone [22, 23, 24].

However, it is far from obvious that the usual notion of entanglement, defined with reference to non-relativistic physics, is an appropriate quantum resource for relativistic systems described by QFT. A proper quantum resource should be compatible with strong locality and causality constraints on acceptable physical observables that are required by QFT. Indeed, Fermi's problem is an indication that special care is needed in identifying acceptable local observables in a relativistic quantum system.

Our model. In this research, we study the causal propagation of information between separated Unruh-DeWitt (UdW) detectors [2], rather than between two atoms. An Unruh-DeWitt detector is a point-like quantum system that interacts with a quantum scalar field through a dipole coupling that mirrors the coupling of atoms to the electromagnetic field. The interaction of multiple UdW detectors with a quantum fields has been employed in order to study, for example, accelerator induced disentanglement [25], entanglement creation outside the lightcone [24], relativistic teleportation protocols [26], causal propagation of signals perturbatively [27], and entanglement harvesting [28].

The main benefit of using the UdW detectors for studying information transfer is that they admit exact solutions. In particular, if (i) the self-Hamiltonian of each detector corresponds to a harmonic oscillator, and (ii) the initial state of the field is Gaussian, then the system of N detectors interacting with the quantum field is mathematically equivalent with a Quantum Brownian Motion (QBM) model for N oscillators in bath. The latter system is exactly solvable [6, 29, 30]. Hence, we can compare the predictions of any approximation with those of the exact solution.

This Ph.D. thesis is structured as follows.

In Chapter II, we present some basic concepts of the theory of open quantum systems. We provide the derivation of the (Markovian) second order master equation–which is used to describe the dynamics of an open system–focusing on the approximations that are usually employed in its derivation.

In Chapter III, we introduce the Quantum Brownian Motion (QBM) model for multipartite systems. We present the general solution to a QBM model with N-system oscillators interacting with an environment at a thermal state.

In Chapter IV, we present our model of two Unruh-DeWitt detectors interacting via a massless scalar field. The two-detectors model is a special case of the N-system QBM model. We find an exact solution to the evolution equations of the two-detector system. We prove that the Markov approximation breaks down completely when describing the transfer of information between remote detectors. In Chapter V, we present some applications of our model. We derive the generalized uncertainty relations for the two-detectors system. Employing the Positive Partial Transpose (PPT) criterion, we identify a unique asymptotic state that is correlated, and entangled at small distances. Moreover, we demonstrate that our two-detector model exhibit the same non-causal behaviour to the Fermi's two atom system, and we discuss the implications, and how causality can be restored.

Finally, in Chapter VI, we summarize and discuss the results of our thesis.

Chapter 2

Open Quantum Systems

In this chapter, first, we present the difference between closed and open quantum systems. Also, we provide the second-order Master equation that describes the dynamics for the case of open quantum systems. Besides, we describe the approximations and the models that we use to construct the Master equation, i.e., the equation of motion for the reduced density matrix of the system.

2.1 Theoretical Background

In the Quantum Information Theory, the qubit is considered the basic unit of organization of information. Unlike the classic bit that can be prepared at only two states, either $|0\rangle$ or $|1\rangle$, the qubit can be prepared in the superposition of state $|\psi(t)\rangle = a|0\rangle + b|1\rangle$, where a, b are complex numbers. Due to this property, i.e., the superposition principle can be used (a) in quantum optics for the study of photon polarization, (b) in the physics of concentrated matter, and (c) in a multitude of other physical systems. A particular interest is (d) its application to quantum information systems.

For the sake of simplicity, we can represent the qubit states as points on the Bloch sphere (Fig. 2.1). The states $|0\rangle$ and $|1\rangle$ are points at the north and south pole. Also, the eigenstates of the Pauli operators σ_x and σ_y corresponds to the x and y axes.

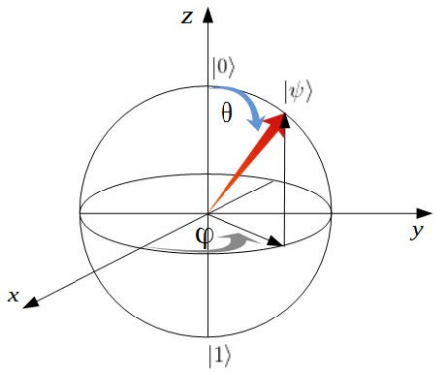


FIGURE 2.1: Representation of a qubit state on the surface of the Bloch sphere.

According to the literature, in order to implement the quantum information systems, we can use the open quantum systems. An open quantum system interacts with the surrounding environment and loses part of its quantum state in the form of energy and information. The loss of energy between system and environment is known as decoherence [31], [32], [33], [5], and is often regarded as the important problem of quantum technologies, as it destroys all quantum resources of the system, such as entanglement and superposition.

Every open quantum system S interacts with a surrounding system that it called environment E. The latter is usually approached by many degrees of freedom. The total $S + E$ system is a closed quantum system. In this access, we can only have to the degrees of freedom specified by the system S. The state of this subsystem changes due to its interaction with the environment, and this development can no longer be described by a Hamiltonian.

The study of open quantum systems requires the evolution of the density matrix, commonly called the Quantum Master Equation. In the literature, for the description of open quantum systems, we usually chose two models, the quantum Brownian motion model [34] and the spin-boson model, The first model consists of one or more particles interacting with a thermal bath. The bath is modeled by a set of harmonic oscillators, initially at a thermal state of temperature T [35].

In the second model, the system interacts with the environment, which is modeled by a set of $\frac{1}{2}$ spin particles. This model is suitable for very low-temperature environments and for well-located ways of oscillating the environment [36]. Therefore, this model can be used very well at the laboratory [37], [38]. A subcategory of the quantum Brownian model is that in which a harmonic oscillator interacts with an environment that is treated as a thermal bath in the equilibrium state and consists of an infinite number of oscillators [34]. This model completed with the construction of the Master equation, but also with the finding the solution of [29], [6] and [39], [40]. In most realistic systems, the Master equation is derived from many approximations. One of them is the Markov approximation, that is considered the most important, according to which, the time evolution of the state of the system becomes in a larger time scale than the time scale that characterizes the environment correlation. The system does not have a memory of its previous state, and it is known as the Markovian system. The interaction of open systems with the environment is suitable for the study of the phenomena such as (a) the loss of energy from the system to the environment, (b) the diffusion, (c) the decoherence [31] and (d) the entanglement.

2.1.1 Open quantum systems: description

Most realistic physical systems interact with their environment and exchange energy and information. For one to study such systems, it should extend the class of closed systems already existing in quantum mechanics by introducing a new, general categorization of *open quantum systems*.

An open quantum system is a quantum system S, which interacts with another quantum system called environment E. [1] The total $S + E$ system is a closed quantum system, and the access we can have is only in the degrees of freedom defined by the system S (Fig: 2.2). The state of this subsystem changes due to its interaction with the environment, and a Hamiltonian can not describe this development. The dynamics of this subsystem are called *reduced system dynamics* and the system *reduced system*.

The shift towards the study of open quantum systems arose from the need to manipulate the transmission of quantum information, which implies the control and the full determination of the environmental impact of the quantum system.

Because the theory of the open quantum systems describes the real physical systems and specifically their interaction with the environment, they are suitable for studying phenomena such as energy loss from the system to the environment, dissipation, entanglement, and decoherence. At this point, it is worth for us to mention that all these phenomena treated by the literature as extremely critical parameters for (i) the advancement of research and (ii) the creation of quantum computing.

In contrast to the study of closed systems, whose time evolution was described by the Schrodinger equation and a unitary operator, the study of open quantum systems requires the equation of density matrix evolution. The equation of evolution of the density matrix is called the *Quantum Master Equation*. It is non-unitary and irreversible, and for her construction, we use some approximations and theoretical models.

An essential model in the study of open quantum systems is the *quantum Brownian motion model*. This model describes one or more particles interacting with a heat bath. The bath is modeled by a large number of harmonic oscillators initially at the thermal state. It is used to describe even non-Markovian systems, i.e., open systems that display memory phenomena. There are two matrices, the *dissipation* and the *noise kernel*, that contain all the information that the open system loses during the interaction with the environment.

2.1.2 The dynamics of closed quantum systems

A quantum system is closed when it does not interchange any information with its surrounding system. The time evolution of a state $|\psi(t)\rangle$ of a closed quantum system is unitary i.e. can be described by a unitary operator U. So, for a closed system with Hamiltonian $\hat{H}(t) = \hat{H}_S \otimes I_E + I_S \otimes \hat{H}_E + \hat{H}_{int}$, is given by the Schrodinger equation:

$$i\frac{d|\psi(t)\rangle}{dt} = \hat{H}(t)|\psi(t)\rangle, \tag{2.1}$$

[1] The system that called environment is usually approached by many degrees of freedom, and for the sake of simplicity it is modeled using spectral density.

where $\hat{H}(t)$ is the Hamiltonian of the system and $\hbar = 1$. The solution of Schrodinger equation is determined by the equation:

$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle. \tag{2.2}$$

where $\hat{U}(t, t_0)$ is the unitary time-evolution operator, that satisfies the condition:

$$\hat{U}^\dagger(t, t_0)\hat{U}(t, t_0) = \hat{U}(t, t_0)\hat{U}^\dagger(t, t_0) = I. \tag{2.3}$$

With the substitution of the eq. (2.2) into eq. (2.1), the unitary time-evolution operator $\hat{U}(t, t_0)$ takes the form:

$$i\frac{\partial}{\partial t}\hat{U}(t, t_0) = \hat{H}(t)\hat{U}(t, t_0), \tag{2.4}$$

with initial condition:

$$\hat{U}(t_0, t_0) = I. \tag{2.5}$$

In the case of a closed isolated system (the Hamiltonian is time independent), the operator $U(t, t_0)$ is given by the equation:

$$\hat{U}(t, t_0) = e^{-i\hat{H}(t-t_0)}. \tag{2.6}$$

However, if Hamiltonian H is time-dependent, the operator $\hat{U}(t, t_0)$ is given by the time-ordered exponential:

$$\hat{U}(t, t_0) = T_{\leftarrow} e^{-i\left[\int_{t_0}^{t} ds\hat{H}(s)\right]} \tag{2.7}$$

where $T_{\leftarrow}$ is the chronological time-ordering operator, which assigns time-dependent operators from right to left in the direction of the arrow. Assuming that the system is not in a pure state $|\psi(t)\rangle$ but in a mixed state, for its description we use an operator $\hat{\rho}$, called a density matrix. The density matrix operator has the following form:

- for a pure state, i.e. a physical state with maximal knowledge

$$\hat{\rho}_0(t) = |\psi_a(t_0)\rangle\langle\psi_a(t_0)|, \tag{2.8}$$

- for a mixed state, i.e. a physical state with partial knowledge

$$\hat{\rho}_0(t) = \sum_a w_a|\psi_a(t_0)\rangle\langle\psi_a(t_0)| \tag{2.9}$$

where $\hat{\rho}$ is a statistical operator and w_a are positive weights.

The time evolution of a mixed state with density matrix $\hat{\rho}(t)$ is called the **Liouville-von Neumann equation** and can be written as:

$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{H}(t), \hat{\rho}(t)] \tag{2.10}$$

where $\hat{\rho}(t)$ can be written as:

$$\hat{\rho}(t) = \hat{U}(t, t_0)\hat{\rho}(t_0)\hat{U}(t, t_0) \tag{2.11}$$

and $\hbar = 1$. The equation (2.10) is the quantum analogue of the **classical Liouville equation**, which is written as:

$$\frac{d\hat{\rho}(t)}{dt} = \hat{L}(t)\hat{\rho}(t) \tag{2.12}$$

where $\hat{L}(t)$ is called Liouville super-operator because it acts on an operator to yield another operator and is equal to $-i[\hat{H}(t), \hat{\rho}(t)]$. The solution of the equation (2.12), in correspondence with the expression (2.7) is the following:

$$\hat{\rho}(t) = T_{\leftarrow} exp\left[\int_{t_0}^{t} ds\hat{L}(s)\right]\hat{\rho}(t_0) \tag{2.13}$$

and in the case of independent Hamiltonian take the form:

$$\hat{\rho}(t) = e^{[\hat{L}(t-t_0)]}\hat{\rho}(t_0). \tag{2.14}$$

2.2 The dynamics of open quantum systems

As we mentioned at the beginning of the chapter, an open system is a quantum S system that interacts with another quantum system E called environment (see Fig. (2.2)). The environment E is usually the system with the infinite number of degrees of freedom, and for its description, particular modeling is used, as well as appropriate initial conditions. The evolution of the subsystem S is due to the interaction of this subsystem with the environment E. This interaction is manifested with two essential processes: one of them is the dissipation, and another is the noise. In the diffusion case, there is a loss of energy from system S to environment E. On the other hand, in the second process, there is a loss of energy which returns to the system again. In this case, however, the system lost any information. Because of these processes, the time evolution of the system S can not be described with unitary operators but obeys a dynamic called reduced system dynamics. In this case, the system is characterized as a reduced quantum system. It follows a schematic representation of an open quantum system.

Both the system S and the environment E are subsystems of a larger, closed, generally quantum system $S + E$. Each individual system is characterized by a Hilbert space. The system S is described by Hilbert H_S, the environment E from H_E and the total system $S + E$ from the $H \otimes H_E$. The total Hamiltonian consists of three terms and is given by the following expression:

$$\hat{H}_{tot} = \hat{H}_S \otimes I_E + I_S \otimes \hat{H}_E + \hat{H}_{int} \tag{2.15}$$

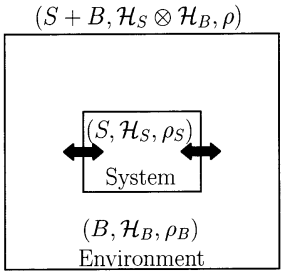


FIGURE 2.2: *Schematic representation of an open quantum system $S + E$, with $E = B$ [33].*

where $\hat{H}_S$ is the self-Hamiltonian of the open system S, $\hat{H}_E$ is the free Hamiltonian of the environment E, and $\hat{H}_{int}$ is the Hamiltonian that describing the interaction between the system and the environment.

The theory of Open Systems (O.Q.S.) describes the dynamics of the evolution of the subsystem S with similar techniques with the statistical mechanics into non-equilibrium quantum systems. An open quantum system is a theoretical construct suitable for the study of the dynamics of the non-equilibrium quantum systems and therefore is very useful for addressing fundamental issues (such as the transition from quantum to classical theory through the environment caused by the phenomenon called decoherence).

As mentioned earlier, an open quantum system always interacts with its environment, and during this interaction, two effects happen. Energy from the system is transferred to the environment (dissipation), or the information is transferred from system to environment and a part of this, returns again to the system. In this case, any information is lost (noise).

The theory of open systems is critical in several fields such as physics of condensed matter, quantum optics [41], the theory of quantum measurement, [42], non-equilibrium field theory, quantum cosmology and in semi-classic gravity.

In most cases, the study of the total system $S + E$ and especially of the environment E with infinite degrees of freedom is a complex issue. It is necessary to construct a set of differential equations describing the dynamics of the total system. Also, sometimes, the determining of the quantum quantities of interest requires finding their expectation value, but it refers to some of the degrees of freedom of the total system. To be able to describe quantum systems in which we do not have full access, we use several approximations to construct a set of physical relevant observables, that is, a level of description or coarse-graining of the system.

If we consider an observable A, that refer to subsystems in the pure state with expactation value given by the form:

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle \tag{2.16}$$

and using the definition of density matrix

$$\rho := |\psi|\rangle\langle\psi|, \tag{2.17}$$

the expectation value of an observable A, can be written as:

$$\langle\hat{A}\rangle = tr_S\{\hat{A}\hat{\rho}_S\} \tag{2.18}$$

where

$$\hat{\rho}_S = tr_E\hat{\rho} \tag{2.19}$$

is the reduced density matrix of the system S, and is obtained by taking the partial trace over to the degrees of freedom of the environment E . There are two classes of state representing by the density matrix $\hat{\rho}$. The first one is the pure state, and the second one is the mixed state. In the first case, all the objects of the ensemble are in the same state, represented by the pure state, and the expectation value can be written as:

$$\langle A\rangle = \sum_n |c_n^2|a_n \tag{2.20}$$

where a_n is the eigenvalue of the Hermitian operator A and $|c_n^2|$ is the probability of the measurement of the a_n. On the other hand, if the objects of the ensemble are not in the same state, then they define the statistical mixture or the mixed-state. So, the reduced density matrix can be calculated as:

$$\hat{\rho}_S(t) = \sum_i p_i|\psi_i(t)\rangle\langle\psi_i(t)| \tag{2.21}$$

and the expectation value as:

$$\langle A\rangle = \sum_i p_i\langle\psi_i|A|\psi_i\rangle \tag{2.22}$$

where $|\psi_i(t)\rangle$ are the orthogonal basis and $\sum_i p_i = 1$. The density matrix has some properties, two of them are an useful tool for the measurement of mixedness, i.e. :

- $Tr\hat{\rho} = 1 \Longrightarrow$ for a pure state

- $Tr\hat{\rho} \leq 1 \Longrightarrow$ for a mixed state.

If we need to determine the dynamics of the system S and not the environment E, which is considered unchanged, we use the reduced density matrix operator $\hat{\rho}_S(t)$. This operator derives from the density matrix of the total system $\hat{\rho}(t)$ by taking the partial trace over the degrees of freedom of the environment, and corresponds to the density matrix $\hat{\rho}(t)$ which obeys the Liouville-von Neumann equation with $\hbar = 1$, i.e. the description of its equation of motion is given by the expression:

$$\frac{d}{dt}\hat{\rho}_S(t) = -itr_E[\hat{H}(t), \hat{\rho}(t)] \tag{2.23}$$

In most cases, we study complex systems in which finding the reduced density matrix with the above process is not an easy task. But using the Master equations, we can directly calculate the density matrix of the system $\hat{\rho}_S(t)$ with the help of the relation:

$$\hat{\rho}_S(t) = V(t)\hat{\rho}(t) \tag{2.24}$$

where the matrix $V(t)$ is a dynamical map and called super-operator, which means that acts to operators.

2.3 Quantum Markovian dynamics

The Markov approximation refers to the small memory that displays the environment when interacting with the system. If we suppose that the interaction between the system S and the environment is weak and if we consider that at the time $t = 0$, the state of the total system can be described by the following equation:

$$\hat{\rho}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_E(0) \tag{2.25}$$

where $\hat{\rho}_S(0)$, $\hat{\rho}_E(0)$ are the initial state of reduced system and the initial state of the environment, respectively. The evolution of the density matrix $\hat{\rho}_S(t)$ from the initial state at time $t = 0$ to another state $\hat{\rho}(t)$ at time t is given by:

$$\hat{\rho}_S(0) \to \hat{\rho}_S(t) \tag{2.26}$$

where the operator $\hat{V}(t)$ is called **super-operator** as it acts on the $S(H_S)$ of the reduced density matrix of the system into self.

$$V(t) : S(H_S) \to S(H_S) \tag{2.27}$$

For the total system, we have:

$$\hat{\rho}(t) = U(t, t_0)\hat{\rho}(t_0)U^{\dagger}(t, t_0) \tag{2.28}$$

therefore

$$\rho_S(t) = tr_E\{U(t, t_0)[\hat{\rho}_S \otimes \hat{\rho}_E(0)]U^{\dagger}(t, t_0)\} \tag{2.29}$$

The above display is called dynamic map and describes the evolution of the state of the open system at time t. For the study of dynamical map, we use the spectral decomposition of the density matrix operator $\hat{\rho}_E$. So we have:

$$\hat{\rho}_E = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i| \tag{2.30}$$

where $|\phi_i\rangle$ is a orthogonal basis into Hilbert space H_E and λ_i are the positive real number, that satisfy the condition:

$$\sum_i \lambda_i = 1 \tag{2.31}$$

The map V_t is a completely positive linear mar if it can be represented by the Kraus representation, which means that it can be described as:

$$\hat{V}(t)\hat{\rho}_S = \sum_{\alpha,\beta} \hat{K}_{\alpha\beta}(t)\rho_S \hat{K}^{\dagger}_{\alpha\beta}(t) \tag{2.32}$$

where,

$$\hat{K}_{\alpha\beta}(t) = \sqrt{\lambda_\beta} < \Phi_\alpha|\hat{U}(t,0)|\Phi_\beta > \tag{2.33}$$

The operators $\hat{K}_{\alpha\beta}(t)$ obey the relation:

$$\sum_{\alpha\beta} \hat{K}^{\dagger}_{\alpha\beta}(t)\hat{K}_{\alpha\beta}(t) = I_S \tag{2.34}$$

So, we get:

$$tr_S\{\hat{V}(t)\hat{\rho}_S\} = tr_S\hat{\rho}_S = 1 \tag{2.35}$$

The dynamical map Φ_t acts at the any initial state $\hat{\rho}_S(0)$ of the system and corresponds the initial state to the state at time t, $\hat{\rho}_S(t)$, according to the following representation:

$$\hat{\rho}_S(0) \mapsto \hat{\rho}_S(t) = \hat{V}_t\hat{\rho}_S(t) \tag{2.36}$$

The dynamical map $\hat{V}_t$ is a completely positive linear map with trace $Tr[\hat{V}_t\hat{\rho}] = Tr\hat{\rho}$. The dynamical map satisfies the following property:

$$\hat{V}(t_1)\hat{V}(t_2) = \hat{V}(t_1 + t_2),\ t_1, t_2 \geq 0 \tag{2.37}$$

2.3.1 Quantum Master Equation: Markovian approximation

In the case where the quantum semi-group has the following form:

$$\hat{V}(t) = e^{\hat{L}t} \tag{2.38}$$

where L is a linear map, then the dynamics of the system can be describe from the first order differential equation, that called **Markovian Quantum**

Master Equation and has the form:

$$\frac{d}{dt}\hat{\rho}_S(t) = \hat{L}\hat{\rho}_S(t) \tag{2.39}$$

The construction of the super-operator $\hat{L}$ requires that the complete basis of the orthogonal operators $\hat{G}_i, i = 1, 2, \cdots, N^2$ can be determined as:

$$(\hat{G}_i, \hat{G}_j) \equiv tr_S\{\hat{G}_i^\dagger \hat{G}_j\} = \delta_{ij} \tag{2.40}$$

with $tr_s\hat{G}_i = 0$. Applying the completeness relation to $\hat{K}_{\alpha\beta}(t)$ operators, we obtain:

$$\hat{K}_{\alpha\beta}(t) = \sum_{i=1}^{N^2} \hat{G}_i(\hat{G}_i, \hat{K}_{\alpha\beta}(t)) \tag{2.41}$$

In this case, the dynamical map takes the form:

$$\hat{V}(t)\hat{\rho}_S = \sum_{i,j=1}^{N^2} c_{ij}(t)\hat{G}_i\hat{\rho}_S\hat{G}_j^\dagger \tag{2.42}$$

where

$$c_{ij}(t) \equiv \sum_{\alpha\beta}(\hat{G}_i, \hat{K}_{\alpha\beta}(t))(\hat{G}_j, \hat{K}_{\alpha\beta}(t))^* \tag{2.43}$$

Finally, we have :

$$\hat{L}\hat{\rho}_S = -i[\hat{H}, \hat{\rho}_S] + \{M, \hat{\rho}_S\} + \sum_{i,j=1}^{N^2} \alpha_{ij}\hat{G}_i\hat{\rho}_S\hat{G}_j^\dagger \tag{2.44}$$

where,

$$\alpha_{ij} = \lim_{\epsilon\to 0} \frac{c_{ij}(\epsilon)}{\epsilon} \tag{2.45}$$

and

$$\hat{M} = -\frac{1}{2}\sum_{i,j=1}^{N^2-1} a_{ij}G_j^\dagger G_i \tag{2.46}$$

If we use the transformation:

$$\hat{G}_i = \sum_{k=1}^{N^2-1} u_{ki}\hat{A}_k \tag{2.47}$$

with

$$u\alpha u^{\dagger} = \begin{pmatrix} \gamma_1 & 0 & \dots & 0 \\ 0 & \gamma_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & \gamma_{N^2-1} \end{pmatrix} \tag{2.48}$$

we have:

$$\hat{L}\hat{\rho}_S = -i[\hat{H}, \hat{\rho}_S] + \sum_{k=1}^{N^2-1} \gamma_k \left(\hat{A}_k \hat{\rho}_S \hat{A}_k^{\dagger} - \frac{1}{2}\hat{A}_k^{\dagger}\hat{A}_k\hat{\rho}_S - \frac{1}{2}\hat{\rho}_S\hat{A}_k^{\dagger}\hat{A}_k \right) \tag{2.49}$$

So, the Master equation can be written as:

$$\frac{d}{dt}\hat{\rho}_S(t) = -i[\hat{H}, \hat{\rho}_S(t)] + D(\hat{\rho}_S(t)) \tag{2.50}$$

where

$$D(\hat{\rho}_S(t)) = \sum_{k=1}^{N^2-1} \gamma_k \left(\hat{A}_k \hat{\rho}_S \hat{A}_k^{\dagger} - \frac{1}{2}\hat{A}_k^{\dagger}\hat{A}_k\rho_S - \frac{1}{2}\hat{\rho}_S\hat{A}_k^{\dagger}\hat{A}_k \right) \tag{2.51}$$

is the dissipator. The equation (2.50) consists of two terms. The first term corresponds to the unitary evolution of the system and is the Von-Neumann equation. The second term describes the non-unitary evolution of the system, and it describes the information needed for dissipation and decoherence. The equation (2.50) is called the Gorini-Kosakowski-Sudarshan-Lindblad (GKSL) Master equation or Lindblad Master Quantum Equation. All information on the system's interaction with the environment is expressed by the second term of the right-hand part of the equation (2.50). The operators $\hat{A}_K$ are often called Lindblad operators, and the parameters γ_k are called relaxation rates and are the ones that together with the Hamiltonian interaction $\hat{H}_{int}$ define the interaction channels with the environment.

We conclude that by working on the Markov approximation, we can construct the quantum Master equation for the system, and it is already known its form. The quantum Master equation in this approximation has the form (2.50), of a Lindblad-type quantum Master equation. In the Markov approximation, which is applied at very high temperatures, during system and environment interaction, the information lost by the system is not saved in the environment, and therefore, the environment does not display memory effects.

2.4 Weak-coupling limit: Born approximation

An important approximation to construct a quantum Master equation is the Born approximation, which works with the Markov approximation, so we are talking about the Born-Markov approximation. The Born approximation

is based on the hypothesis of weak coupling approximation between system and environment. In this approximation, the environment or the reservoir as we might otherwise find it is a heat bath. It is in thermal equilibrium, and in the case of this equilibrium perturbation, the environment returns in a concise time. The time to return to equilibrium, i.e., the response time in the Born approximation, is very short.

2.4.1 Redfield equation

If we consider the the interacting between the system S and environment is weak, the total Hamiltonian has the form:

$$\hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_I \tag{2.52}$$

where, $\hat{H}_S$, $\hat{H}_E$ are the Hamiltonian of the system and environment and $\hat{H}_I$ is the Hamiltonian of interaction between the system and the environment. The evolution of the total system in the interaction picture, is given by:

$$\rho(t) = e^{(H_S+H_E)t}\rho(t)e^{-i(H_S+H_E)t} \tag{2.53}$$

The von-Neumann equation in the interaction picture for the total density matrix is:

$$\frac{d}{dt}\hat{\rho}(t) = -i[\hat{H}_I(t), \hat{\rho}(t)] \tag{2.54}$$

where

$$\hat{\rho}(t) = \hat{\rho}(0) - i\int_0^t ds[\hat{H}_I(s), \hat{\rho}(s)] \tag{2.55}$$

The equations (2.54) and (2.55) yield:

$$\frac{d}{dt}\hat{\rho}_s(t) = -\int_0^t dstr_E[\hat{H}_I(t), [\hat{H}_I(s), \hat{\rho}(s)]] \tag{2.56}$$

We suppose that:

$$tr_E[\hat{H}_I(t), \hat{\rho}(0)] = 0 \tag{2.57}$$

The equation (2.56) corresponds to the equation of motion for the density matrix $\hat{\rho}(t)$ of the total system. Using the Born approximation, which the coupling of the system and the environment is negligible and therefore the effect of the system on the environment is very small (weak-coupling approximation), without implying that the system cannot cause any kind of stimulation in the environment. Therefore, due to this negligible effect of the system on the environment, the density matrix of the total system can be written as follows:

$$\hat{\rho}(t) \approx \hat{\rho}_S(t) \otimes \hat{\rho}_E(t) \tag{2.58}$$

and the equation (2.56) becomes:

$$\frac{d}{dt}\hat{\rho}_S(t) = -\int_0^t dstr_E[\hat{H}_I(t),[\hat{H}_I(s),\hat{\rho}_S(s)\otimes\hat{\rho}_E]] \tag{2.59}$$

The above equation can be simplified using the Markov approximation, in which the state of the system refers to the present and it does not depends on the past history. Therefore the Master equation takes the following form:

$$\frac{d}{dt}\hat{\rho}_S(t) = -\int_0^t dstr_E[\hat{H}_I(t),[\hat{H}_I(s),\hat{\rho}_S(t)\otimes\hat{\rho}_E]] \tag{2.60}$$

This equation is called Redfield equation, and although local in time, it is not a Markovian quantum Master equation since it depends on the initial state of the equation and substitution of $s \to t-s$ is sufficient. So, we have:

$$\frac{d}{dt}\hat{\rho}_S(t) = -\int_0^\infty dstr_E[\hat{H}_I(t),[\hat{H}_I(t-s),\hat{\rho}_S(t)\otimes\hat{\rho}_E]] \tag{2.61}$$

Now, we must introduce the Hamiltonian of interaction. If we assume the general diagonal form such as:

$$\hat{H}_I(t) = \sum_\alpha \hat{A}_\alpha \otimes \hat{B}_\alpha \tag{2.62}$$

where $\hat{A}_\alpha^\dagger = \hat{A}_\alpha$ and $\hat{B}_\alpha^\dagger = \hat{B}_\alpha$, the equation (2.61) becomes:

$$\frac{d}{dt}\hat{\rho}_S(t) = -\int_0^\infty ds \sum_{\alpha\beta} tr_E[\hat{A}_\alpha(t)\otimes\hat{B}_\alpha(t),[\hat{A}_\beta(t-s)\otimes\hat{B}_\beta(t-s),\hat{\rho}_S(t)\otimes\hat{\rho}_E]]$$

to

(2.64)

We define the operators:

$$\hat{A}_\alpha(\omega) \equiv \sum_{\epsilon-\epsilon'} \Pi(\epsilon)\Pi(\epsilon') \tag{2.65}$$

with the following properties:

$$[\hat{H}_S,\hat{A}_\alpha(\omega)] = -\omega\hat{A}_\alpha(\omega) \tag{2.66}$$

$$\left[\hat{H}_S,\hat{A}_\alpha^\dagger(\omega)\right] = \omega\hat{A}_\alpha^\dagger(\omega) \tag{2.67}$$

$$e^{i\hat{H}_S t}\hat{A}_\alpha(\omega)e^{-i\hat{H}_S t} = e^{-i\omega t}\hat{A}_\alpha(\omega) \tag{2.68}$$

$$e^{i\hat{H}_S t}\hat{A}_\alpha(\omega)e^{-i\hat{H}_S t} = e^{i\omega t}\hat{A}_\alpha^\dagger(\omega) \tag{2.69}$$

Also,

$$[\hat{H}_S, \hat{A}^\dagger_\alpha(\omega)\hat{A}_\beta(\omega)] = 0 \quad (2.70)$$

$$\hat{A}^\dagger_\alpha(\omega) = \hat{A}_\alpha(-\omega) \quad (2.71)$$

$$\sum_\omega \hat{A}_\alpha(\omega) = \sum_\omega \hat{A}^\dagger_\alpha(\omega) = \hat{A}_\alpha \quad (2.72)$$

Finally, the Hamiltonian can be written as:

$$\hat{H}_I = \sum_{\alpha,\omega} \hat{A}_\alpha(\omega) \otimes \hat{B}_\alpha = \sum_{\alpha,\omega} \hat{A}^\dagger_\alpha(\omega) \otimes \hat{B}^\dagger_\alpha \quad (2.73)$$

In the interaction picture:

$$\hat{H}_I = \sum_{\alpha,\omega} \hat{A}_\alpha(\omega) \otimes \hat{B}_\alpha = \sum_{\alpha,\omega} \hat{A}^\dagger_\alpha(\omega) \otimes \hat{B}^\dagger_\alpha \quad (2.74)$$

where

$$\hat{B}_\alpha(t) = e^{i\hat{H}_E t}\hat{B}_\alpha e^{-i\hat{H}_E t} \quad (2.75)$$

with:

$$tr_E[\hat{H}_I(t), \hat{\rho}(0)] = 0 \quad (2.76)$$

and finally,

$$< \hat{B}_\alpha(t) > \equiv tr\{\hat{B}_\alpha(t)\rho_E\} = 0 \quad (2.77)$$

the equation (2.61) becomes:

$$\frac{d}{dt}\hat{\rho}_S(t) = \int_0^\infty ds tr_E\{\hat{H}_I(t-s)\hat{\rho}_S(t)\hat{\rho}_E\hat{H}_I(t) - \hat{H}_I(t)\hat{H}_I(t-s)\hat{\rho}_S(t)\hat{\rho}_E\} + h.c. \quad (2.78)$$

$$= \sum_{\omega,\omega'}\sum_{\alpha,\beta} e^{i(\omega-\omega')t}_{\alpha\beta}(\omega)(\hat{A}_\beta(\omega)\hat{\rho}_S(t)\hat{A}^\dagger_\alpha(\omega') - \hat{A}^\dagger_\alpha(\omega')\hat{A}_\beta(\omega)\hat{\rho}_S(t)) + h.c. \quad (2.79)$$

where

$$C_{\alpha\beta}(\omega) \equiv \int_0^\infty ds e^{i\omega s} < \hat{B}^\dagger_\alpha(t)\hat{B}_\beta(t-s) > \quad (2.80)$$

Introducing the environment correlation functions, with the relation:

$$< \hat{B}^\dagger_\alpha(t)\hat{B}_\beta(t-s) > \equiv tr_E\{\hat{B}^\dagger_\alpha(t)\hat{B}_\beta(t-s)\hat{\rho}_E\} \quad (2.81)$$

and assuming that the environment is in thermal equilibrium, that is:

$$[\hat{H}_E, \hat{\rho}_E] = 0 \quad (2.82)$$

$$< \hat{B}^\dagger_\alpha(t)\hat{B}_\beta(t-s) > = < \hat{B}^\dagger_\alpha(t)\hat{B}_\beta(0) > \quad (2.83)$$

We have:

$$\frac{d}{dt}\hat{\rho}_S(t) = \sum_{\omega}\sum_{\alpha\beta} C_{\alpha\beta}(\omega)(\hat{A}_\beta(\omega)\hat{\rho}_S(t)\hat{A}^\dagger_\alpha(\omega) - \hat{A}^\dagger_\alpha(\omega)\hat{A}_\beta(\omega)\hat{\rho}_S(t)) + h.c. \quad (2.84)$$

where

$$C_{\alpha\beta}(\omega) = \frac{1}{2}\gamma_{\alpha\beta}(\omega) + iS_{\alpha\beta}(\omega) \quad (2.85)$$

and

$$S_{\alpha\beta}(\omega) = \frac{1}{2i}(C_{\alpha\beta}(\omega) - C^*_{\alpha\beta}(\omega)) \quad (2.86)$$

$$\gamma_{\alpha\beta} = C_{\alpha\beta}(\omega) + C^*_{\alpha\beta}(\omega) = \int_0^\infty ds e^{i\omega s}\langle \hat{B}^\dagger_\alpha(t)\hat{B}_\beta(0)\rangle \quad (2.87)$$

Finally, the Master equation in the interaction picture is as follows:

$$\frac{d}{dt}\hat{\rho}_S(t) = -i[\hat{H}_{LS}, \hat{\rho}_S(t)] + D(\hat{\rho}_S(t)) \quad (2.88)$$

with

$$\hat{H}_{LS} = \sum_{\omega}\sum_{\alpha\beta} S_{\alpha\beta}(\omega)\hat{A}^\dagger_\alpha(\omega)\hat{A}_\beta(\omega) \quad (2.89)$$

and

$$D(\hat{\rho}_S(t)) = \sum_{\alpha\beta}\gamma_{\alpha,\beta}\gamma_{\alpha\beta}(\omega)\left(\hat{A}_\beta(\omega)\hat{\rho}_S(t)\hat{A}^\dagger_\alpha(\omega) - \frac{1}{2}\{\hat{A}^\dagger_\alpha(\omega)\hat{A}_\beta(\omega), \hat{\rho}_S\}\right) \quad (2.90)$$

The coefficients $\gamma_{\alpha\beta}$ is the relaxation rates for the different decay modes of the open system and are given in the terms of the correlation functions of the environment. Also, this coefficients contain all the information about the energy and information that losed of the system during its interaction with the environment. The RWA [33] approach should also be used to convert the above equation to Lindblad form. Using this approximation also results:

$$\frac{d}{dt}\hat{\rho}_S(t) = -i[\hat{H}_{LS}, \hat{\rho}_S(t)] - \frac{1}{2}\sum k_\mu k_\mu [\hat{L}_\mu, [\hat{L}_\mu, \hat{\rho}_S(t)]] \quad (2.91)$$

All the physics is at the coefficients k_μ and most of the time it's a difficult task to calculate.

At this point, it is worth noting that the description of the dynamics of the reduced quantum system using the Markovian quantum master equations for time scale greater than the order of magnitude of the correlation time t_E has not yet been resolved ($t_S \gg t_E$).

2.5 Rotating Wave Approximation

The Rotating Wave Approximation (R.W.A.) is a useful approximation for the construction of the Lindblad Master equation and is divided into two categories. The first one, which we call **pre-trace rotating wave (pre-TRWA)**, is the approach applied before we trace out the degrees of freedom of the environment and which results in a new Hamiltonian, free from all its terms Hamiltonian interaction oscillate very fast. The second category that distinguishes R.W.A. is called **post-TRWA**, refers to the conversion of the Master equation for the open quantum system, to a Lindblad Master equation, and is applied after tracing out the degrees of environmental freedom in the Master equation. According to the report [11] for a detailed description of the quantum state of an open quantum system and the calculation of entanglement dynamics, the R.W.A. approach is not recommended. In particular, in the case of a thermal environment in equilibrium state or multi-frequency environment, only post-TRWA gives valid results. Therefore, the R.W.A. approximation is not a useful approximation for finding accurate results, especially in the case of qubit study over a very long distance.

2.6 Perturbation theory

The Master equation is a differential equation describing the evolution of the reduced density matrix of an open system. The Master equation for linear coupling to the high-temperature ohmic environment was first developed by Caldeira-Leggett [34], expanded by Unruh and Zurek [43] and eventually emerged for the general environment (i.e., arbitrary spectral density function) by Hu, Paz, and Zhang [6]. This effect can be extended to the case of nonlinear coupling using perturbation theory by leading only second-order terms.

In quantum mechanics, the perturbation theory is a set of approximations used to describe a complex system with a simpler matrix. Using the solutions mentioned in the Hamiltonian of the simple system, we can derive the solutions for many complex quantum systems. Since it is an approximation method, we can use it to construct a Master equation, that is, an equation describing the time evolution of an open quantum system. The perturbation theory applies only in the case of weak system-environment interaction, and in most cases, we hold conditions up to second order. Finally, mainly in quantum optics, one more approach, the Wigner-Weisskopf approach, is used to study phenomena that require the construction of a Master equation.

The approximations mentioned above allow us to construct the Master equation more easily. In most models, the Master equations are of the second order Lindblad type, and the system-environment interaction is expressed through the A_k operators of the equation (2.51). Although this equation gives us valid results for the evolution of an open quantum system, since it has been extracted using approximations, it has no general validity. It can be applied in the case of high temperatures and when the interaction between the

system and the environment is weak. It cannot be applied at low temperatures if the quantum system under study is degenerate and when we are interested in studying quantum entanglement.

Therefore, for the study of open quantum systems in which non-Markovian phenomena occur, and therefore the second-order Master equation cannot be applied, we use higher-order Master equations or theoretical models such as the quantum Brownian model and the spin-boson model. In the case of open quantum systems, the quantum Brownian motion model is mainly used. We describe this model in more detail in the next section.

2.6.1 System of one harmonic oscillator: Caldeira-Leggett Model

In 1981 Amir Caldeira and Anthony J. Leggett proposed a simple quantum model for the studying of the local dissipation for the high temperature environment. This model consists of a Brownian quantum particle in one dimension that interacts with a high temperature thermal reservoir [34]. This particle has a mass of m and a position of x and is described by the following free Hamiltonian:

$$\hat{H}_S = \frac{1}{2}\hat{p}^2 + V(x) \tag{2.92}$$

where p is the momentum of the particle. The thermal bath with which the particle interacts consists of a set of harmonic oscillators, with frequencies ω_n and mass m_n and is described by the Hamiltonian:

$$\begin{aligned} \hat{H}_E &= \sum_n \omega_n \left(\alpha_n \alpha_n^\dagger + \frac{1}{2} \right) \\ &= \sum_n \left(\frac{1}{2m_n}\hat{p}_n^2 + \frac{1}{2} m_n \omega_n^2 \hat{x}_n^2 \right) \end{aligned} \tag{2.93}$$

where $\alpha_n^\dagger$, α_n are the creation and annihilation operators of the environment, and x_n, p_n are the corresponding coordinates and canonical conjugate momentum. The Hamiltonian interaction is as follows:

$$\hat{H}_I = -\hat{x} \sum_n k_n \hat{x}_n \equiv -\hat{x}\hat{B} \tag{2.94}$$

where

$$\hat{B} = \sum_n k_n \hat{x}_n = \sum_n k_n \sqrt{\frac{\hbar}{2m_n\omega_n}} \left(\hat{\alpha}_n + \hat{\alpha}_n^\dagger \right) \tag{2.95}$$

The Hamiltonian of the interaction has the form:

$$\begin{aligned}\hat{H} &= \hat{H}_S + \hat{H}_E + \hat{H}_I + \hat{H}_C \\ &= \frac{1}{2m}\hat{p}^2 + V_C(x) + \sum_n \left(\frac{1}{2m_n}\hat{p}_n^2 + \frac{1}{2}m_n\omega_n^2\hat{x}_n^2\right) - \hat{x}\sum_n k_n\hat{x}_n \end{aligned} \quad (2.96)$$

where

$$V_C(x) = V(x) + x^2\sum_n \frac{k_n^2}{2m_n\omega_n^2} \quad (2.97)$$

and

$$\hat{H}_C = x^2\sum_n \frac{k_n^2}{2m_n\omega_n^2} \quad (2.98)$$

The term H_C is called counter-term and it applies only to the Hilbert space H_S of the particle. To study the motion equation of this particle, a Master equation must be found. At the weak coupling limit and at high temperature the Markovian Master equation is called Master Caldeira-Leggett [34] equation, which corresponds to the Markovian regime and has the form:

$$\frac{d}{dt}\hat{\rho}_S(t) = -i[\hat{H}_S, \hat{\rho}_S(t)] - i\gamma[\hat{x}, \{\hat{p}, \hat{\rho}_S(t)\}] - 2m\gamma k_B T[\hat{x}, [\hat{x}, \hat{\rho}_S(t)]] \quad (2.99)$$

The equation (2.99) was introduced by Caldeira-Leggett for the case where the environment is ohmic and using the spectral density $J(\omega)$, is defined as:

$$J(\omega) = \sum_k \frac{k_n^2}{2m_n\omega_n}\delta(\omega - \omega_n) \quad (2.100)$$

and for $\omega \to 0$, takes the form:

$$J(\omega) = \frac{2m\gamma}{\pi}\omega \quad (2.101)$$

The equation (2.99) consists of three terms, with a different physical interpretation. The first term in the right part of the equation describes the unitary evolution of its system dynamics. The second term, which is proportional to the relaxation constant γ, corresponds to the energy loss due to the system's interaction with the environment. The last term is proportional to temperature and is the one describing the diffusion phenomena observed when interacting with the environment. The diffusion term describes the temperature fluctuations and is fundamental to the theoretical description of the decoherence.

The Master Caldeira-Leggett equation is a Markovian master equation but it is not of the Lindblad type. This equation can be converted to Lindblad by adding a term which, at the high temperature limit, is small. Therefore,

by adding this term, we obtain the following Lindblad Master equation:

$$i\partial_t\hat{\rho} = [\hat{H}_0, \hat{\rho}] - \gamma[\hat{x}, \hat{\rho}\hat{P} + \hat{P}\hat{\rho}] - 2iM\gamma T[\hat{X}, [\hat{X}, \hat{\rho}]] \tag{2.102}$$

The eq. (2.102) consists of three terms: the first term corresponding to unitary evolution, the second term describing energy losses, and the last one describing the diffusion term.

2.7 Non-Markovian dynamics

In previous sections, we present the Markovian master equations that only applied if the environmental memory effects can be ignored. However, in many cases (e.g., in a low-temperature environment), the Markovian Master equation is not satisfied, and there is a strong dependence of the evolution of the system on its capture in the past times. So, we must introduce the non-Markovian master equations [44]. These equations are not local in time and are generally difficult to solve in detail. For any non-Markovian Master equation

$$\frac{d\hat{\rho}(t)}{dt} = \int_0^t \hat{K}(t-\tau)\rho(\tau) \tag{2.103}$$

with the propagator $\hat{G}(t)$: $\rho(0) \to \rho(t)$, corresponds a Master equation of the form

$$\frac{d\hat{\rho}(t)}{dt} = \hat{K}(t)\hat{\rho}(\tau) \tag{2.104}$$

where $\hat{K}(t) = \hat{G}(t)\hat{G}(t)^{-1}$. This equation is time non-local and it can be determined via Laplace transformation. I.e., there are many other cases where non-Markovian equations are local in time and they are in the form of Markovian differential equations.

$$\frac{d\hat{\rho}(t)}{dt} = \hat{K}(t)\hat{\rho}(t) \tag{2.105}$$

where the $\hat{K}(t)$ is a super-operator, that depends explicitly on time (but not from earlier times, which makes this equation local in time). The eq. (2.105) is a linear first-order differential equation for the open system state. In the next chapter, we will see a case of a local time non-Markovian equation master (e.g. quantum Brownian motion).

The study of open quantum systems requires the construction of the equation of motion, i.e., a Master equation. Because this construction is not easy, we use different approximations or theoretical models for the understanding of the non-Markovian dynamics. By using the above methods, we end up constructing a Lindblad Master Equation. The quantum Brownian motion

model is a theoretical model for the description of the loss of energy and decoherence, even in the case of the studying of the non-Markovian systems. In this model, all information consist of two functions, which we will see in detail in the next section, the dissipation kernel and the noise kernel contain all the information needed to describe open quantum systems. In the following section, we describe the formalism of the quantum Brownian motion model (Q.B.M.), and we construct the Master equation using the Wigner function.

Chapter 3

Quantum Brownian motion model for general case

In this chapter, we begin by defining the formalism of the quantum Brownian motion model for the case of multipartite systems [29]. Then, we construct the homogeneous equation of motion of our system, and we determine the expansion of two matrices that containing all physics of the open system. The matrices called dissipation and noise kernel. After the constructing of the equations of motion and the calculation of these matrices, we calculate the corresponding expansions for the simple model of a system of one harmonic oscillator.

For the study of open quantum systems, it is important to define a Hilbert space H_{S+E}, which includes system S and environment E, and it determines the level of description. One of the most common models in open systems theory, which is very useful in quantum theory, quantum optics, and decoherence, is the Quantum Brownian motion model. One of the advantages of quantum Brownian motion model is that they can accurately describe many physical processes. According to this model, the selected degrees of freedom, that is, what we call a system, are described by the Hilbert space H_S, and the remaining degrees of freedom (environment) by the Hilbert space $\hat{H}_E$. It is described by the Hilbert space H_{S+E}. The time evolution of the whole system is described by Schrodinger's equation with Hamiltonian H. This Hamiltonian is a sum of three terms: the Hamiltonian of the system H_{sys}, the Hamiltonian of the environment $\hat{H}_{env}$ and the Hamiltonian of interaction $\hat{H}_{int}$.

Specifically, the study of open quantum systems requires the description of the time evolution of the **reduced density matrix** of the system. The reduced density matrix is defined by the partial trace of the density density $\hat{\rho}(t)$ of the total system, with the relation $\hat{\rho}_S(t) = Tr_E\hat{\rho}(t)$. The most general method of producing the Master equation is that of [6] where they used route basic techniques and, most importantly, the Feynman-Vernon path integral.

For describing the time evolution of an open quantum system, we must construct a Master equation. If we assume that, at the initial time, the system and the environment are separable and not interacting, in the SchrÃűdinger picture, the density matrix can be written as a direct product of the density matrix of the system and the density matrix of the environment. I.e.

$$\hat{\rho}(t_0) = \hat{\rho}_{syst}(t_0) \otimes \hat{\rho}_{env}(t_0) \tag{3.1}$$

Working at the Heisenberg picture, we assume that the density matrix at the time t is equal with the density matrix at the time t_0,

$$\hat{\rho} = \hat{\rho}_{syst} \otimes \hat{\rho}_{env} \tag{3.2}$$

If the density matrix is known for a set of degrees of freedom in the representation position, we can always define the corresponding Wigner function, which contains precisely the same information. The reduced Wigner function is similar to the distribution function in the phase space coordinates, although it is not always defined positively. The equation that describes the time evolution of the Wigner function is similar to the Fokker-Planck equation for classical statistical systems.

The most important and useful tool of the study of the open quantum systems is the Wigner function propagator, which in the case of linear systems, is a δ function[45]. It is, therefore, possible that the effects of the environment on the system are divided into two categories: unitary and non-unitary. In the case of linear systems, the Wigner function propagator can be calculated just for arbitrary temperatures and is Gaussian. In the case of systems with more general dynamics, it can not be calculated precisely, except where a semi-classic approach can be used.

Using the Wigner function, we have:

$$W(\mathbf{X},\mathbf{P}) = \frac{1}{(2\pi)^N}\int d\zeta e^{-\imath\,\mathbf{P}\cdot\zeta}\hat{\rho}\left(\mathbf{X}+\frac{1}{2}\zeta,\mathbf{X}+\frac{1}{2}\zeta\right) \tag{3.3}$$

with inverse:

$$\hat{\rho}(\mathbf{X},\mathbf{Y}) = \int dP e^{\imath\mathbf{P}\cdot(\mathbf{X}-\mathbf{X}')}W\left(\frac{1}{2}(\mathbf{X}+\mathbf{X}'),\mathbf{P}\right) \tag{3.4}$$

3.1 Time evolution in QBM models

The time evolution for initial state in QBM models, defined by the equation[29] :

$$\widehat{\rho}_t(\mathbf{X}_f,\mathbf{Y}_f) = \int d^N X_0 d^N Y_0 J(\mathbf{X}_f,\mathbf{Y}_f,t|\mathbf{X}_0,\mathbf{Y}_0,0)\widehat{\rho}_0(\mathbf{X}_0,\mathbf{Y}_0), \tag{3.5}$$

where $J(\mathbf{X}_f,\mathbf{Y}_f,t|\mathbf{X}_0,\mathbf{Y}_0,0)$: is the density matrix propagator and it is independent of initial state.

The Wigner function propagator $K(\mathbf{X}_f,\mathbf{P}_f,t|\mathbf{X}_0,\mathbf{P}_0,0)$ can be defined as:

$$\begin{aligned} K\left(\mathbf{X}_f,\mathbf{P}_f,t|\mathbf{X}_0,\mathbf{P}_0,0\right) &= \int \frac{d\zeta_f d\zeta_0}{(2\pi)^N} e^{\imath\mathbf{P}_0\cdot\zeta_0-\imath\mathbf{P}_f\cdot\zeta_f} \\ &\times J\left(\mathbf{X}_f+\frac{\zeta_f}{2},\mathbf{X}_f-\frac{\zeta_f}{2},t|\mathbf{X}_0+\frac{\zeta_0}{2},\mathbf{X}_0-\frac{\zeta_0}{2},0\right) \end{aligned} \tag{3.6}$$

with the coordinates:

$$\begin{aligned} \xi_a &= (X_1, X_2, \cdots X_N, P_1, P_2, \cdots, P_N), \\ a &= 1, 2, \cdots, 2N, \end{aligned} \tag{3.7}$$

the Wigner function propagator $K_t(\xi_f, \xi_0)$ can be written as:

$$W_t = \int \frac{d^{2N}\xi_0}{(2\pi)^N} K_t(\xi_f, \xi_0)\, W_0(\xi_0) \tag{3.8}$$

where W_0 and W_t is the Wigner function for the time $t = 0$ and for the time t, respectively.

The Wigner function propagator in the case of QBM models is the **Gaussian**, and defined as follow:

$$K_t(\xi_f, \xi_0) = \frac{\sqrt{detS^{-1}}}{\pi^N} \times exp\left[-\frac{1}{2}[\xi_f^a - \xi_{cl}^a(t)]S_{ab}^{-1}(t)[\xi_f^b - \xi_{cl}^b]\right], \tag{3.9}$$

where ξ_{cl}^a,ξ_{cl}^b are the solutions of classical equation of motion for the system variables and can be calculated with the equation $\xi_{cl}^a = R_b^a(t)\xi_0^a$. Also the matrix $S_{ab}^{-1}(t)$ is the positive and it is important for the construction of Master equation.

3.1.1 Covariance matrix

The Wigner function propagator as we can see from the eq. (4.6,4.7) depends on the matrices $R(t)$, $S(t)$. To find this matrices, we start by calculating by the covariance matrix V_t. As we will see follow, this matrix contains both the matrix $R(t)$ and $S(t)$.

The two-point correlation matrix V, can be defined by the following equation:

$$V_{ab} \overset{def}{=} \frac{1}{2}Tr[\hat{\rho}(\hat{\xi}_a\hat{\xi}_b + \hat{\xi}_b\hat{\xi}_a)] - Tr(\hat{\rho}\hat{\xi}_a)Tr(\hat{\rho}\hat{\xi}_b) \tag{3.10}$$

Using the equations (4.6) and (4.7), the two-point correlation matrix V for the time t, can be written as [1]

$$V_t = R(t)V_0R^T(t) + S(t) \tag{3.11}$$

where V_0 is the correlation matrix of the initial state and $S(t)$ is the correlation function.

The eq. (3.11) has two terms. The term $R(t)V_0R^T(t)$ of the equation (3.11) expresses the time evolution of the correlation functions according to the classical equation of motion. The second term is independent of the initial state and describes the information of the interaction between the system and the environment. The fact that the matrix S(t) does not depend on the initial state

[1] In Appendix C there are a detailed description of the proof of the equation 4.9.

indicates to us, how to determine it, we only use the part of the correlation matrix that is independent of the initial state.

3.2 The system of N harmonic oscillators

3.2.1 Quantum Brownian models

We consider a system of N harmonic oscillators of masses M_r and frequencies Ω_r interacting with a heat bath. The bath is modelled by a set of harmonic oscillators of masses m_i and frequencies ω_i. The Hamiltonian of the total system can be written as:

$$\hat{H} = \hat{H}_{syst} + \hat{H}_{env} + \hat{H}_{int} \tag{3.12}$$

where

$$\hat{H}_{syst} = \sum_r \left(\frac{1}{2M_r} \hat{P}_r^2 + \frac{M_r \Omega_r^2}{2} \hat{X}_r^2 \right) \tag{3.13}$$

$$\hat{H}_{env} = \sum_i \left(\frac{1}{2m_i} \hat{p}_i^2 + \frac{m_i \omega_i^2}{2} \hat{q}_i^2 \right) \tag{3.14}$$

$$\hat{H}_{int} = \sum_i \sum_\alpha c_{ir} \hat{X}_r \hat{q}_i = \hat{X}_r \otimes \sum_i c_i \hat{q}_i \equiv \hat{X}_r \otimes \hat{E} \tag{3.15}$$

3.2.2 The evolution of the oscillators of the bath

The time evolution of the oscillators of the environment can be de described with following equations, according to [29]:

$$\ddot{\hat{q}}_i(t) + \omega_i^2 \hat{q}_i(t) = \sum_r \frac{c_{ir}}{m_i} \hat{X}_r(t) \tag{3.16}$$

$$\hat{q}_i(t) = \hat{q}_i^0(t) + \sum_r \frac{c_{ir}}{m_i \omega_i} \int_0^t ds \sin[\omega_i(t-s)] \hat{X}_r(s) \tag{3.17}$$

where

$$\hat{q}_i^0(t) = \hat{q}_i cos(\omega_i t) + \frac{\hat{p}_i}{m_i \omega_i} \sin(\omega_i t) \tag{3.18}$$

.

For the oscillators of the system, the equation of motion is defined as:

$$\ddot{\hat{X}}_r(t) + \Omega_r^2 \hat{X}_r(t) + \frac{2}{M_r} \sum_r \int_0^t ds \gamma_{rr'}(t-s) \hat{X}_{r'}(s) = \sum_i \frac{c_{ir}}{M_r} \hat{q}_i^0(t) \tag{3.19}$$

where

$$\gamma_{rr'}(s) = -\sum_i \frac{c_{ir} c_{ir'}}{2 m_i \omega_i^2} \sin(\omega_i s), \tag{3.20}$$

is the dissipation kernel.

The solution of Eq.(3.19) is :

$$\hat{X}_r(t) = \sum_{r'} \left(\dot{u}_{rr'} \hat{X}_{r'} + \frac{1}{M_{r'}} u_{rr'} \hat{P}_r \right) + \sum_{r'} \frac{1}{M_{r'}} \int_0^t ds u_{rr'}(t-s) \sum_i c_{ir} \hat{q}_i^0(s). \quad (3.21)$$

The solution of the homogeneous part of equation (3.19), with initial conditions $u_{rr'}(0) = 0$ and $\dot{u}_{rr'}(0) = \delta_{rr'}$ is:

$$u(t) = L^{-1}[A_{rr'}^{-1}(z)] \quad (3.22)$$

where

$$A_{rr'}^{-1}(z) = (z^2 + \Omega_r^2) + \frac{2}{M_r} \tilde{\gamma}_{rr'}(z) \quad (3.23)$$

and $\tilde{\gamma}_{rr'}(z)$ is the Laplace transform of the dissipation kernel. Therefore, for the construction of the matrix $R(t)$, first of all we must to calculate the dissipation kernel for our system, then the inverse Laplace of this and finally the solution of classical equation of motion $u(t)$.

3.2.3 Equations of correlation functions of harmonic oscillators

In the case we have a system of harmonic oscillators in a thermal temperature T, the correlation functions $S(t)$ can be writen as [29]:

$$S_{X_r X_{r'}} = \sum qq' \frac{1}{M_q M_{q'}} \int_0^t ds \int_0^t ds' u_{rq}(s) v_{qq'}(s-s') u_{q'r'}(s'), \quad (3.24)$$

$$S_{P_r P_{r'}} = M_r M_{r'} \sum_{qq'} \frac{1}{M_q M_{q'}} \int_0^t ds \int_0^t ds' \dot{u}_{rq} v_{qq'}(s-s') \dot{u}_{q'r'}(s'), \quad (3.25)$$

$$S_{X_r P_{r'}} = M_{r'} \sum_{qq'} \frac{1}{M_q M_{q'}} \int_0^t ds \int_0^t ds' u_{rq}(s) v_{qq'}(s-s') \dot{u}_{q'r'}(s') \quad (3.26)$$

where the symetric matrix $v_{rr'}(s)$ is the noise kernel, and can be defined as

$$v_{rr'}(s) = \sum_i \frac{c_{ir} c_{ir'}}{2 m_i \omega_i^2} \coth\left(\frac{\omega_i}{2T}\right) \cos(\omega_i s) \quad (3.27)$$

3.2.4 The Master equation for a system of N harmonic oscillators

The Wigner function can be written as:

$$W_t(\xi) = \int \frac{d^{2N}\xi_0}{(2\pi)^N} K_t(\xi_f, \xi_0) W_0(\xi_0) \quad (3.28)$$

where W_t and W_0 is the Wigner functions at times 0, t. In Quantum Brownian motion models, if the system has quadratic Hamiltonian and the initial state for the environment is Gaussian, the Wigner function propagator is Gaussian and can be written as

$$K_t(\xi_f,\xi_0)=\frac{\sqrt{detS^{-1}(t)}}{\pi^N}\times\exp\left[-\frac{1}{2}\left[\xi_f^a-\xi_{cl}^a(t)\right]S_{ab}^{-1}(t)\left[\xi_f^b-\xi_{cl}^b(t)\right]\right] \quad (3.29)$$

The equations 3.28 and 3.29, yield:

$$\begin{aligned} W_t(\xi) &= \int\frac{d^{2N}\xi_0}{(2\pi)^N}\frac{\sqrt{detS^{-1}(t)}}{\pi^N} \\ &\times \exp\left[-\frac{1}{2}\left[\xi_f^a-\xi_{cl}^a(t)\right]S_{ab}^{-1}(t)\left[\xi_f^b-\xi_{cl}^b(t)\right]\right]W_0(\xi_0). \end{aligned} \quad (3.30)$$

Using the identities

$$\int\frac{d^{2N}\xi_0}{2\pi^N}\left(\xi-\xi_{cl}\right)^a K_t\left(\xi_f-\xi_0\right)W_0(\xi_0)=-S^{ab}\frac{\partial W_t(\xi)}{\partial\xi^b}, \quad (3.31)$$

$$\int\frac{d^{2N}\xi_0}{(2\pi)^N}\left(\xi-\xi_{cl}\right)^a\left(\xi-\xi_{cl}\right)^b K_t(\xi_f,\xi_0)W_0(\xi_0)=S^{ab}+S^{ac}S^{bd}\frac{\partial^2 W_t(\xi)}{\partial\xi^c\partial\xi^d}, \quad (3.32)$$

finally, we have:

$$\frac{\partial W_t}{\partial t}=-\left(\dot{R}R^{-1}\right)_b^a\frac{\partial(\xi^b W_t)}{\partial\xi^a}+\left(\frac{1}{2}\dot{S}^{ab}-(\dot{R}R^{-1})_c^{(a}S^{cb)}\right)\frac{\partial^2 W_t(\xi)}{\partial\xi^a\partial\xi^b} \quad (3.33)$$

The above Eq. (3.33) is the Master equation for the system and is a sum of two terms. The first of them corresponds to the dissipation terms, and describes the irreversible process that take place in homogeneous systems. Finally, the second one corresponds to the diffusion terms.

In summary, the recipe for deriving the coefficients of the master equation for QBM models in an environment is a follow. First, we compute the dissipation and noise kernel; second, we solve the classical equations defined by the elementary functions $u_{rr'}(s)$. In the next chapter, we calculate the exact solutions to the open system dynamics for the system of two harmonic oscillators, and we show that the system of two harmonic oscillators is a special case of the Q.B.M. for the N-system oscillators.

Chapter 4

Exact solutions to the open system dynamics

In this chapter, we present the general solution to the QBM model with N-system oscillators interacting via an environment, and show that for the case of the system of two detectors interacting through a scalar field is a special solution. Also, we find the explicit solution to the two-detector system, and we prove that the Markov approximation breaks down completely for the transmission of information between two remote detectors.

4.1 The model

4.1.1 QBM in a multi-partite system

The Hamiltonian

We consider a system of N harmonic oscillators of masses M_α and frequencies Ω_α interacting with a heat bath. The bath is modeled by a set of harmonic oscillators of masses m_i and frequencies ω_i. The Hamiltonian of the total system is

$$\hat{H} = \hat{H}_{syst} + \hat{H}_{env} + \hat{H}_{int} \tag{4.1}$$

where

$$\hat{H}_{syst} = \sum_\alpha \left(\frac{1}{2M_\alpha} \hat{P}_\alpha^2 + \frac{M_\alpha \Omega_\alpha^2}{2} \hat{X}_\alpha^2 \right), \tag{4.2}$$

$$\hat{H}_{env} = \sum_i \left(\frac{1}{2m_i} \hat{p}_i^2 + \frac{m_i \omega_i^2}{2} \hat{q}_i^2 \right), \tag{4.3}$$

$$\hat{H}_{int} = \sum_i \sum_\alpha c_{i\alpha} \hat{X}_\alpha \hat{q}_i, \tag{4.4}$$

where $c_{i\alpha}$ are coupling constants.

Since the total Hamiltonian is quadratic concerning all positions and momenta, the evolution operator $e^{-i\hat{H}t}$ can be explicitly constructed, and its position matrix elements are Gaussian.

We consider a factorized initial condition $\hat{\rho}_{sys} \otimes \hat{\rho}_{env}$ for the total system. If $\hat{\rho}_{env}$ is Gaussian, then the reduced density matrix propagators can be computed explicitly. For $N = 1$, the reduced dynamics leads to the Hu-Paz-Zhang master equation [6].

In general, the assumption of a factorized initial condition between field and detectors is meaningful only as far as the field modes with energies of the order of the frequencies Ω_α is concerned. There is no preparation that can enforce separability for photons at the infra-red and ultra-violet edges of the spectrum. However, a non-factorized initial condition does not allow us to consider general initial states for the field [46] in many model systems, including QBM, the effect of the non-factorizing initial state die out after a time-scale of the order of a high-frequency cut-off [47].

The Wigner function propagator

In this paper, we will employ the solution to the multi-partite QBM model in the Wigner representation [48, 49, 29]—another form of the general solution is found in [30]. The Wigner function for the reduced density matrix is defined as

$$W(\mathbf{X}, \mathbf{P}) \;=\; \frac{1}{(2\pi)^N} \int d\zeta e^{-\imath\, \mathbf{P}\cdot\zeta} \hat{\rho}\left(\mathbf{X} + \frac{1}{2}\zeta, \mathbf{X} - \frac{1}{2}\zeta\right). \tag{4.5}$$

We use the coordinates $\xi^a = (X_1, X_2, \ldots, X_N, P_1, P_2, \ldots, P_N)$ on phase space; the Wigner function is expressed as $W(\xi)$. Dynamics in the Wigner picture is implemented by the Wigner function propagator $K_t(\xi_f, \xi_0)$, namely, a kernel that evolves the initial Wigner function W_0 to the Wigner function W_t at time t,

$$W_t(\xi_f) = \int \frac{d^{2N}\xi_0}{(2\pi)^N} K_t\left(\xi_f, \xi_0\right) W_0(\xi_0). \tag{4.6}$$

For QBM models, the Wigner function propagator is Gaussian. The most general form of a Gaussian propagator is

$$K_t(\xi_f, \xi_0) \;=\; \frac{\sqrt{det S^{-1}}}{\pi^N}\, exp\left[-\frac{1}{2}[\xi_f^a - \xi_{cl}^a(t)] S^{-1}_{ab}(t) [\xi_f^b - \xi_{cl}^b(t)]\right]. \tag{4.7}$$

where S_{ab} is positive definite matrix and

$$\xi_{cl}^a(t) = R_b^a(t)\xi_0^b. \tag{4.8}$$

The matrix R_b^a defines the solution to the classical equations of motion. The matrix S_{ab} determines the evolution of the environment-induced fluctuations. To see this, we consider the correlation matrix

$$V_{ab} := \frac{1}{2} Tr[\hat{\rho}(\hat{\xi}_a\hat{\xi}_b + \hat{\xi}_b\hat{\xi}_a)] - Tr(\hat{\rho}\hat{\xi}_a) Tr(\hat{\rho}\hat{\xi}_b). \tag{4.9}$$

By Eq. (4.7),

$$V(t) = R(t)V(0)R^T(t) + S(t) \tag{4.10}$$

where V_0 is the correlation matrix of the initial state.

The explicit form of the matrices R and S was derived in [29]. They depend on two kernels, the *dissipation kernel*,

$$\gamma_{\alpha\alpha'}(s) = -\sum_i \frac{c_{i\alpha}c_{i\alpha'}}{2m_i\omega_i^2} sin(\omega_i s), \tag{4.11}$$

and the *noise kernel*,

$$\nu_{\alpha\alpha'}(s) = \sum_i \frac{c_{i\alpha}c_{i\alpha'}}{2m_i\omega_i^2} coth\left(\frac{\omega_i}{2T}\right) cos(\omega_i s). \tag{4.12}$$

The crucial step in the determination of the matrices R and S is to find the solution to the linear integro-differential equation [29]

$$\ddot{u}_\alpha(t) + \Omega_r^2 u_\alpha(t) + \frac{2}{M_\alpha}\sum_{\alpha'}\int_0^t ds\gamma_{\alpha\alpha'}(t-s)u_{\alpha'}(s) = 0 \tag{4.13}$$

with initial conditions $\dot{u}_{\alpha\alpha'}(0) = \delta_{\alpha\alpha'}$ and $u_{\alpha\alpha'}(0) = 0$. Eq. (4.13) is essentially the classical equation of motion with a non-local-in-time dissipation term defined by the dissipation kernel.

Given the solution $u(t)$, we define the matrix R as

$$R = \left(\begin{array}{c|c} \dot{u}(t) & u(t)M^{-1} \\ \hline M\ddot{u}(t) & M\dot{u}(t)M^{-1} \end{array}\right) \tag{4.14}$$

where $M = diag(M_1, ..., M_N)$ is the mass matrix for the system.

The matrix elements of S are given by

$$S_{X_\alpha X_{\alpha'}} = \sum_{\beta\beta'}\frac{1}{M_\beta M_{\beta'}}\int_0^t ds\int_0^t ds' u_{\alpha\beta}(s)\nu_{\beta\beta'}(s-s')u_{\beta'\alpha'}(s'), \tag{4.15}$$

$$S_{P_\alpha P_{\alpha'}} = M_\alpha M_{\alpha'}\sum_{\beta\beta'}\frac{1}{M_\beta M_{\beta'}}\int_0^t ds\int_0^t ds'\dot{u}_{\alpha\beta}(s)\nu_{\beta\beta'}(s-s')\dot{u}_{\beta'\alpha'}(s'), \tag{4.16}$$

$$S_{X_\alpha P_{\alpha'}} = M_{\alpha'}\sum_{\beta\beta'}\frac{1}{M_\beta M_{\beta'}}\int_0^t ds\int_0^t ds' u_{\alpha\beta}(s)\nu_{\beta\beta'}(s-s')\dot{u}_{\beta'\alpha'}(s') \tag{4.17}$$

4.1.2 Two UdW detectors

We consider a system of two identical static harmonic oscillators of mass $M = 1$ and frequency Ω interacting with a scalar field through the Unruh-DeWitt interaction Hamiltonian. The Hamiltonian of the total system form,

where we assume that the detectors are localized at $\boldsymbol{x} = \boldsymbol{x}_1$ and $\boldsymbol{x} = \boldsymbol{x}_2$

$$\hat{H}_{int} = \lambda \left(\int d^3x \hat{\phi}(\boldsymbol{x}) \hat{q}_1 \delta^3(\boldsymbol{x} - \boldsymbol{x}_1) + \int d^n x \hat{\phi}(x) \hat{q}_2 \delta^3(\boldsymbol{x} - \boldsymbol{x}_2) \right). \tag{4.18}$$

where λ is a coupling constant.

For a free scalar field, the total Hamiltonian

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{\omega_k}} (\hat{a}(k) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{a}^\dagger(k) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}), \tag{4.19}$$

is a special case of the QBM Hamiltonian. The index i corresponds to three momenta $\boldsymbol{k}$, $m_i = 1$, $\omega_{\boldsymbol{k}} = |\boldsymbol{k}|$ and $c_{\boldsymbol{k}_\alpha} = \frac{\lambda}{\sqrt{2\omega_k}} e^{ikx_\alpha}$.

It is straightforward to evaluate the dissipation kernel. By Eq. (4.11),

$$\gamma(s) = \gamma_0(s) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \gamma_r(s) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.20}$$

where

$$\gamma_0(s) = -\frac{\lambda^2}{8\pi^2} \int_0^\infty dk sin(ks) \tag{4.21}$$

$$\gamma_r(s) = -\frac{\lambda^2}{8\pi^2 r} \left[\int_0^\infty dk \frac{sin(kr)\sin(ks)}{k} \right]. \tag{4.22}$$

The function $\gamma_0(s)$ is the dissipation kernel of the one-detector system that has been extensively studied in the literature [34]. It must be regularized, for example, by introducing a high-frequency cut-off Λ. For $r \to 0$, γ_r coincides with γ_0. In principle, we should introduce the same cut-off Λ to γ_r, however γ_r is little affected unless r is of the order of Λ^{-1} or smaller. Alternatively, we can regularize γ_0 be equating it with γ_{r_0} for some $r_0 << r$.

By Eq. (4.12), the noise kernel is

$$\nu(s) = \nu_0(s) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \nu_r(s) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.23}$$

where

$$\nu_0(s) = \frac{\lambda^2}{8\pi} \delta(s) \tag{4.24}$$

$$\nu_r(s) = \nu_{21}(s) = \frac{\lambda^2}{32\pi r} \left[\mathrm{sgn}(r-s) + \mathrm{sgn}(r+s) \right]. \tag{4.25}$$

4.2 The classical equations of motion

4.2.1 The inverse Laplace transform

Next, we evaluate the solutions $u_{\alpha\alpha'}(t)$ of the classical equations of motion (4.13). Since Eq. (4.13) is linear, it can be solved by a Laplace transform. It is straightforward to evaluate the Laplace transform $\tilde{u}(z)$ of $u(t)$ as $A^{-1}(z)$, where $A(z)$ is the 2×2 matrix with elements

$$A_{\alpha\alpha'}(z) = (z^2 + \Omega_\alpha^2)\delta_{\alpha\alpha'} + 2\widetilde{\gamma}_{\alpha\alpha'}(z), \tag{4.26}$$

where $\tilde{\gamma}_{\alpha\alpha'}(z)$ is the Laplace transform of the dissipation kernel. The Laplace transforms of γ_0 and γ_r are

$$\tilde{\gamma}_0(z) = -\frac{\lambda^2}{16\pi^2}\ln\left(1+\frac{\Lambda^2}{z^2}\right) \simeq -\frac{\lambda^2}{8\pi^2}\ln\left(\frac{\Lambda}{z}\right) \tag{4.27}$$

$$\tilde{\gamma}_r(z) = -\frac{\lambda^2}{16\pi rz}[e^{-rz}\bar{E}i(rz) - e^{rz}Ei(-rz)], \tag{4.28}$$

where we simplified $\gamma_0(z)$ by assuming that the relevant values of z satisfy $|z| << \Lambda$; Ei stands for the exponential integral function, defined by [50]

$$Ei(z) = \gamma + \ln z + \sum_1^\infty \frac{z^n}{n!n} \tag{4.29}$$

where γ is the Euler-Mascheroni constant.

Then, we obtain

$$\begin{aligned}\tilde{u}(z) &= \frac{1}{2}\left[\frac{1}{z^2+\Omega^2+2\tilde{\gamma}_0(z)+2\tilde{\gamma}_r(z)}\begin{pmatrix}1&1\\1&1\end{pmatrix}\right.\\ &+ \left.\frac{1}{z^2+\Omega^2+2c\tilde{\gamma}_0(z)-2\tilde{\gamma}_r(z)}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\right]\end{aligned} \tag{4.30}$$

Hence, $u(t)$ takes the form,

$$u(t) = \frac{1}{2}\left[f_+(t)\begin{pmatrix}1&1\\1&1\end{pmatrix} + f_-(t)\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\right], \tag{4.31}$$

in terms of the functions $f_\pm(t)$ that is defined by the Bromwich integrals

$$f_\pm(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz \frac{e^{zt}}{z^2+\Omega^2+2\tilde{\gamma}_0(z)\pm 2\tilde{\gamma}_r(z)}. \tag{4.32}$$

The integrated functions in (4.32) have a branch cut at $z = 0$. For this reason, we integrate the function $\frac{e^{zt}}{z^2+\Omega^2+2\tilde{\gamma}_0(z)\pm 2\tilde{\gamma}_r(z)}$ along the contour of Fig.

1 that circles around the branch cut.

Using Cauchy's theorem, we find that the functions $f_\pm(t)$ consists of two parts,

$$f_\pm(t) = f^0_\pm(t) + I_\pm(t). \tag{4.33}$$

The part $f^0_\pm(t)$ contains the contribution from the poles in the region enclosed by the contour, as in Fig. 4.1—we will refer to it as the *pole term*. The part $I_\pm(t)$ includes the contribution from the negative imaginary axis; we refer to this term as the *branch-cut term* [51], [52].

4.2.2 The pole term

For sufficiently small λ, the poles can be identified perturbatively. To this end, we set $z^\pm_+ = \pm i\Omega + \lambda^2 x$, and we solve the equation

$$z^2 + \Omega^2 + 2\tilde{\gamma}_0(z) \pm 2\tilde{\gamma}_r(z) = 0 \tag{4.34}$$

to leading order in λ^2. We find that the poles associated to f_+ are at $z^\pm_+ = \pm i\Omega + i\delta\Omega_+ - \Gamma_+$ and the poles associated to f_- at $z^\pm_- = \pm i\Omega + i\delta\Omega_- - \Gamma_-$, where

$$\delta\Omega_\pm = -\frac{\lambda^2}{8\pi^2\Omega}\left(\ln\left(\frac{\Lambda}{\Omega}\right) \pm \frac{\cos(r\Omega)}{r\Omega}Si(r\Omega) \mp \frac{\sin(r\Omega)}{r\Omega}Ci(r\Omega)\right) \tag{4.35}$$

$$\Gamma_\pm = \Gamma_0\left(1 \pm \frac{\sin(r\Omega)}{r\Omega}\right) \tag{4.36}$$

$$\Gamma_0 = \frac{\lambda^2}{16\pi\Omega}. \tag{4.37}$$

The constant Γ_0 is the decay rate of a single oscillator interacting with a scalar field.

Besides the two poles above, there exists a pole that is not accessible by perturbation theory. This solution corresponds to the regime $|z| << \Omega$. For example, as $r \to \infty$, so that the contribution of the $\tilde{\gamma}_r(z)$ term is negligible, Eq. (4.34) has a root for Re$z \simeq \Lambda e^{-8\pi^2\Omega^2/\lambda^2}$. This root is positive, and it leads to runaway solutions, i.e., it induces a term in $u(t)$ that blows up exponentially as $t \to \infty$. This term is unphysical, as it is incompatible with the dissipative nature of the open system evolution. Its analog appears in the Abraham-Lorentz classical treatment of radiation reaction that leads to a third-order equation for a particle's position [53]. In fact, the exponentially runaway solution in this system was first found by Planck [54]. For the appearance of such terms in QBM models of particle field interaction, see, Ref. [55].

These runaway solutions originate from the inadequacy of the particle-field coupling to account for soft photons. In the present context, runaway solutions can be avoided by an infra-red regularization. For example, we can regularize by assuming a finite mass μ for the scalar field. This is equivalent,

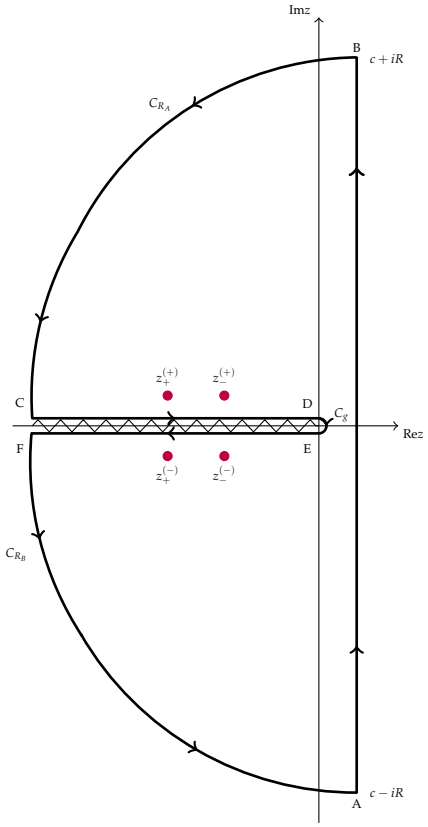


FIGURE 4.1: Bromwich contour, branch cut and poles related to Eq. (4.32). Integration is along a straight line from $c - i\infty$ to $c + i\infty$, where c is a real constant larger than the real part of the poles of the integrand. The contour is closed by a semicircle of radius $R \to \infty$.

to shifting the zero of $\gamma_0(z)$ by μ, so that we redefine

$$\gamma_0(z) = -\frac{\lambda^2}{16\pi^2} \ln\left(1 + \frac{\Lambda^2}{(z+\mu)^2}\right). \tag{4.38}$$

For $\mu > \Lambda e^{-8\pi^2\Omega^2/\lambda^2}$, the third pole has a negative real part and does not lead to runaway solutions. A result of this regularization is that the integrating function manifests branch cuts at $z = -\mu \pm i\Lambda$, which have to be taken into account by appropriate modification of the contour integral. In the weak-coupling limit ($\Gamma_0/\Omega << 1$), μ^{-1} is much larger, and Λ^{-1} is much smaller than physically relevant time-scales, so we can simply ignore the contribution of this pole at the intermediate regimes.

We conclude that in the weak-coupling limit, the pole term is well approximated by, except at very early times ($t \sim \lambda^4$).

$$f_{\pm}^{(0)}(t) = \frac{\sin \tilde{\Omega}_{\pm} t}{\tilde{\Omega}_{\pm}} e^{-\Gamma_{\pm} t}. \tag{4.39}$$

4.2.3 The branch-cut term

To evaluate the integral along the negative near axis, we use the following identities.

$$\tilde{\gamma}_0(-s \pm i\epsilon) = F(s) \mp i\frac{\lambda^2}{16\pi} \tag{4.40}$$

$$\tilde{\gamma}_r(s \pm i\epsilon) = G(s) \mp i\frac{\lambda^2}{16\pi sr}\sinh(rs), \tag{4.41}$$

for positive $\epsilon \to 0$. The functions $F(s)$ and $G(s)$ are

$$F(s) = -\frac{\lambda^2}{8\pi^2}\ln\left(\frac{\Lambda}{s}\right) \tag{4.42}$$

$$G(s) = -\frac{\lambda^2}{8\pi^2 rs}\left[\cosh(rs)\mathrm{Shi}(rs) - \sinh(rs)\mathrm{Chi}(rs)\right] \tag{4.43}$$

where Shi is the hyperbolic sine integral function and Chi the hyperbolic cosine integral function are defined by

$$\mathrm{Shi}(z) = \int_0^t \frac{\sinh(t)}{t}dt, \tag{4.44}$$

$$\mathrm{Chi}(z) = \gamma + lnz + \int_0^z \frac{\cosh(t)-1}{t}dt \tag{4.45}$$

Then,

$$I_\pm(t) = -\frac{\lambda^2}{8\pi^2}\int_0^\infty ds e^{-st}\frac{1 \pm \frac{sinh(rs)}{rs}}{(s^2+\Omega^2+2F(s)+2G(s))^2 + \left(\frac{\lambda^2}{8\pi}\right)^2\left(1 \pm \frac{sinh(rs)}{rs}\right)^2} \tag{4.46}$$

The function $I_\pm(t)$ cannot be evaluated exactly. A good approximation that is valid for $t > r$ is to ignore the terms of order λ^2 in the denominator, so that

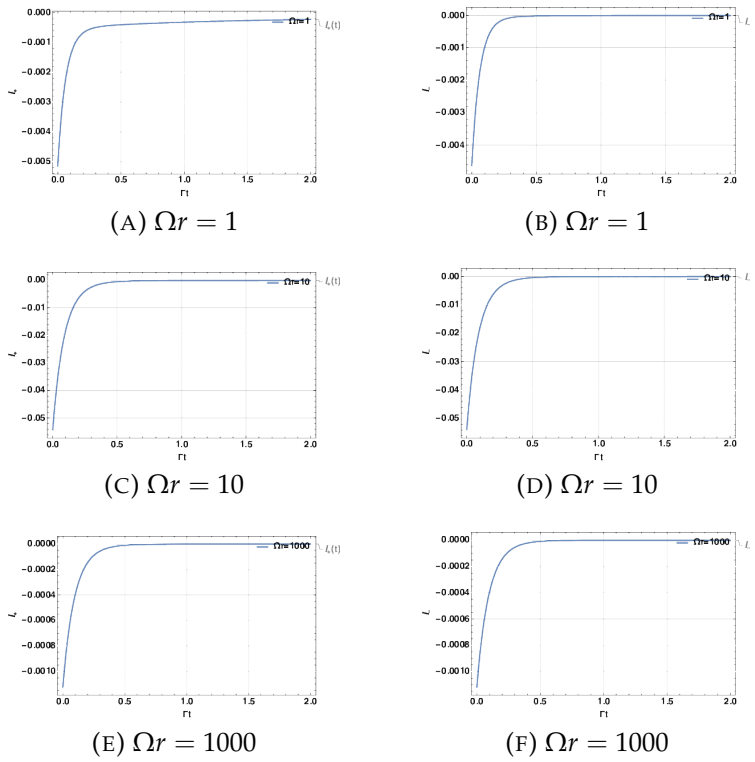
$$I_\pm(t) = -\frac{\lambda^2}{8\pi^2}\int_0^\infty ds e^{-st}\frac{1 \pm \frac{\sinh(rs)}{rs}}{(s^2+\Omega^2)^2} \tag{4.47}$$

For $t < r$, the approximation above does not hold, because dropping the terms of order λ^2 in the denominator renders the integral divergent.

For $\Omega t >> 1$, Eq. (4.47) becomes

$$I_\pm(t) = -\frac{\lambda^2}{8\pi^2\Omega^4}\left[\frac{1}{t} \pm \frac{1}{r}\tanh^{-1}(r/t)\right]. \tag{4.48}$$

In Fig.(4.2) we see the evolution of the function $I_\pm$ as a function of Γt for different values of Ωr. It is negative-valued and increases asymptotically to



(A) $\Omega r = 1$ (B) $\Omega r = 1$

(C) $\Omega r = 10$ (D) $\Omega r = 10$

(E) $\Omega r = 1000$ (F) $\Omega r = 1000$

FIGURE 4.2: Evolution of $\Omega I_{\pm}$ as a function of $\Gamma_0 t$ for different values of Ωr, where $\Gamma_0/\Omega = 10^{-3}$.

zero. It is unlike the pole term, in that it does not involve any oscillations.

4.2.4 The Markov approximation

Eq. (4.32) is similar for the equation for the persistence amplitude of an unstable quantum state in the random phase approximation [7]. In fact, the two kernels $\tilde{\gamma}_0$ and $\tilde{\gamma}_r$ are similar to the ones that appear in the evolution of a pair of atomic qubits interacting with the EM field [56]. The difference is that the dominant term contains a quadratic rather than a linear term with respect to z, reflecting that in a harmonic oscillator, we consider both positive frequency and negative frequency solutions.

The split (4.33) into a pole term and a branch-cut term are generic whenever the kernels describing the effect of the environment contain branch-cuts. A common approximation in the study of unstable systems is the Wigner-Weisskopf approximation (WWA), in which (i) the branch-cut term is neglected, and (ii) the poles are calculated to leading-order in perturbation theory. The WWA approximation leads to exponential decay. It coincides with the van-Hove limit, namely, taking the limit $\lambda \to 0$, with $\lambda^2 t$ kept constant. In the open quantum system context, the van Howe limit leads to the second-order master equation that describes Markovian dynamics [5].

It is straightforward to evaluate the van Howe limit of Eq. (4.32). We can write any function of the form

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz \frac{e^{zt}}{z^2+\Omega^2+\lambda^2 a(z)}, \tag{4.49}$$

for some kernel $\lambda^2 a(z)$, as

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{dz}{i\sqrt{\Omega^2+2\lambda^2 a(z)}}\left[\frac{1}{z-i\sqrt{\Omega^2+2\lambda^2 a(z)}}+\frac{1}{z-i\sqrt{\Omega^2+2\lambda^2 a(z)}}\right] \tag{4.50}$$

We set $z = i\Omega + \lambda^2 x$ in the first term and $z = -i\Omega + \lambda^2 x$ in the second. Then, we take the limit $\lambda \to 0$, with $\lambda^2 t$ constant, to obtain

$$f(t) = \frac{1}{\Omega}\left(e^{-i\Omega t - \frac{\lambda^2 a(i\Omega)}{\Omega}t} - e^{i\Omega t - \frac{\lambda^2 a(-i\Omega)}{\Omega}t}\right), \tag{4.51}$$

i.e., the pole term with a perturbative evaluation of the poles.

The van Howe limit essentially substitutes the classical equation of motion with non-local in time dissipation, with an equation that is local in time. Hence, it removes memory effects from the evolution equation. A local-in-time equation for dissipation is necessary (but usually not a sufficient condition) for Markovian dynamics. This can be seen in path integral derivations of the QBM master equation [34, 6]; Markovian behavior requires that the noise kernel also becomes local.

To summarize, the Markov approximation to the system under study presupposes the validity of the WWA approximation. Hence, its violation is a definite sign of the existence of non-Markovian dynamics.

4.2.5 Non-Markovian dynamics

The WWA, and consequently, the exponential decay law, cannot be valid at all times. Exponential decay fails at very early times due to quantum Zeno dynamics [1]. It also fails at very late times: the branch cut term typically falls off as an inverse power of t, and eventually becomes larger than the pole term that decays exponentially. However, the time scale for this decay is much larger than relaxation time. For example, in optical systems, even for Γ/Ω as large as 10^{-3}, the breakdown of the exponential decay takes place at $\Gamma t \sim 30$, when less than $1 : 10^{26}$ of the initial systems remains in the excited state.

A violation of the WWA is physically meaningful only if it takes place at time-scales compatible with the dissipation time, i.e., if it happens when Γt is a small number. We will show that this takes place in the system studied here when the detectors are separated by a large distance r.

[1] **Quantum Zeno dynamics** called the breakdown of the unitary time evolution at the quantum systems due to the interaction with the environment or the measurement.

Eq. (4.31) implies that $u_{11} = u_{22} = \frac{1}{2}(f_+ + f_-)$ and that $u_{12} = u_{21} = \frac{1}{2}(f_+ - f_-)$. The terms u_{11} and u_{12} describe the dependence of the variables of one detector to the initial conditions of the second detector, while u_{12} and u_{21} essentially describe the correlations developed between the two detectors.

Eq. (4.35, 4.36) imply that as $r \to \infty$, $\Gamma_+ = \Gamma_-$ and $\delta\Omega_+ = \delta\Omega_-$. By Eq. (4.39), $f_+^{(0)}(t) = f_-^{(0)}(t)$ as $r \to \infty$, for all t. Hence, the pole part of $u_{12}(t)$ vanishes for all t as $r \to \infty$. In contrast, the branch-cut term remains finite. By continuity, for any given t there is a finite distance r, at which the branch cut term dominates over the pole term, and hence, the WWA fails.

We have verified this behavior numerically, as can be seen in Fig. (5.3). There, we present a semi-logarithmic plot of the pole term in u_{12} divided by the full u_{12}, as a function of time. By construction, this term is very close to zero if the WWA applies and differs significantly from 0 if the WWA fails. The plots show that the behavior of this function changes when r becomes of the order of Γ_0^{-1}. At this scale, we see significant violations of the WWA at the scale of $\Gamma t \sim 1$, and a complete breakdown as Γt becomes about 5. Note that both violations and the breakdown of the WWA occur early when a significant fraction of energy remains in the system.

The WWA is well preserved for u_{11} and u_{22} at the regime where it fails for u_{12}. Nonetheless, it also fails at sufficiently large times. This is to be expected because—as mentioned earlier— the WWA is guaranteed to fill in the long-time limit. What is rather unexpected, is that for sufficiently large r, the WWA breaks down even for u_{11} and u_{22} at relatively early times. We found that for $\Gamma r < 10$, the breakdown of the WWA occurs at $\Gamma t \simeq 15$, i.e., at a time where a negligible amount of energy remains on the system. However, for $\Gamma r > 50$, the WWA breaks down at $\Gamma t \simeq 5$.

In all regimes that we have studied, the WWA breaks down at the u_{12} term both earlier and more strongly than it does at the u_{11} and u_{22} terms. Therefore, its primary failure is for terms that describe the creation of the correlation between distant detectors. For these terms, the branch-cut contribution dominates. Hence, the creation of correlations over large distances is a non-perturbative effect. It cannot be described correctly by perturbative approximation schemes, such as the von-Hove limit or the second-order master equation.

The conclusion above is unquestionable for the present model, because we have an exact solution, and consequently, full control over all approximation schemes. However, the open system evolution of the oscillator detector should not be significantly different from that of a N-level system coupled to a scalar field. For this reason, we expect that our conclusion is relevant to all systems with a similar Hamiltonian, in particular, to atoms coupled with the electromagnetic field. We have to move beyond the second-order master equation to describe the dynamics of entangled atoms if these atoms are found at separations r of the order of Γ^{-1}.

The system is also non-Markovian at the opposite regime $r \to 0$, as $\gamma_r \to \gamma_0$, and f_- becomes simply $\frac{1}{\Omega}\sin\Omega t$. This behavior has been extensively studied in multi-partite QBM models, see, for example, [57, 58]. We will not be concerned with this regime here, because of the limit $\Omega r << 1$ is not

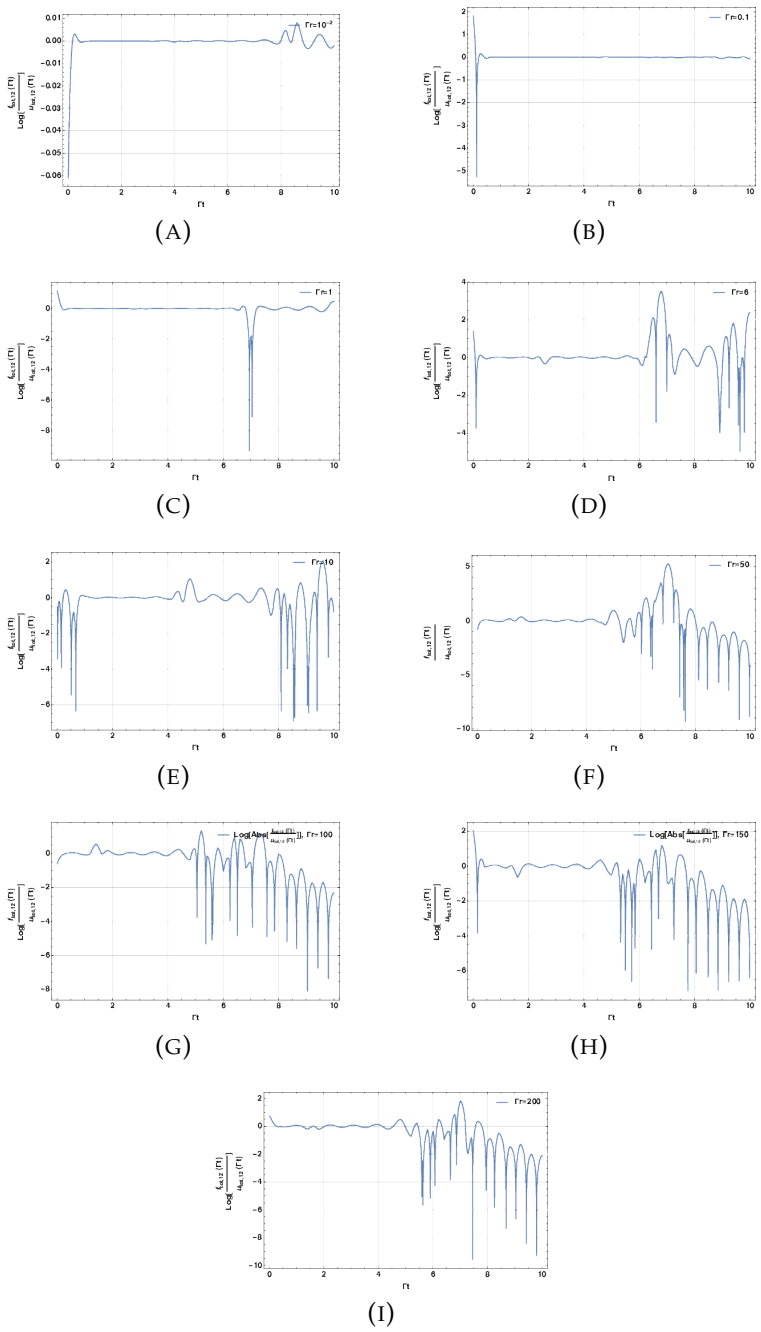


FIGURE 4.3: Evolution of the quantity $\frac{f_{tot}[\Gamma t]}{u_{tot}[\Gamma t]}$ for the non-diagonal elements of the solution $u(t)$ as a function of Γt and for different values of Γr. In this plot $\Gamma/\Omega = 10^{-3}$.

compatible with either the identification of the oscillators with atoms or with particle detectors.

Chapter 5

Applications

In this chapter, we are going to introduce some applications of our exact solvable model. The first application is related to the generalized uncertainty relations. We derive generalized uncertainty relations for our model described in Sec. 4, and we continue with the discussion of the entanglement generation and the challenge of causality.

5.1 General Uncertainty Relations

Within the conception of quantum mechanics, Heisenberg (1927) introduces a mathematical inequality according to which, it is impossible to measure simultaneous and with precise the position and momentum coordinates. This relation called uncertainty relation or Heisenberg uncertainty relation and took the approximate form:

$$\Delta\hat{q}\Delta\hat{p} \geq \frac{\hbar}{2} \tag{5.1}$$

where $\hbar$ is the reduced Planck constant. Kennard-Robertson expected this inequality to an arbitrary pair of operators $\hat{X}$, $\hat{P}$

$$(\Delta\hat{X})^2(\Delta\hat{Y})^2 \geq \frac{1}{4} |< [\hat{X},\hat{Y}] >|^2 \tag{5.2}$$

where $(\Delta\hat{X})$,$(\Delta\hat{Y})$ is the variances (dispersions) of the observables $\hat{X}, \hat{Y}$.

Definition 1. Heisenberg- Kennard- Robertson inequality *If A,B are an arbitrary pair of operators and ΔA ,ΔB are the uncertainties (precisions) of the measurements of two quantum observables, then*

$$(\Delta\hat{A})^2(\Delta\hat{B})^2 - \hat{C}_{AB}^2 \geq \frac{1}{2} \mid Tr(\hat{\rho}[\hat{A},\hat{B}]) \mid^2 \tag{5.3}$$

$$(\Delta\hat{A})(\Delta\hat{B}) \geq \frac{1}{2} \mid Tr(\hat{\rho}[\hat{A},\hat{B}]) \mid \tag{5.4}$$

The uncertainties relation make a statement about the preparation of a quantum state. In special case of the operators $\hat{X}$,$\hat{P}$ the eq.(1) becomes

$$(\Delta\hat{X})(\Delta\hat{P}) \geq \frac{1}{2} \tag{5.5}$$

The eq. (5.1) it seems to be the same with the Heisenberg uncertainty relation but it is not. In eq. (1) the precisions $(\Delta\hat{X})$ and $(\Delta\hat{P})$ is the uncertainties of the measurements of two quantum observables at two different experiment but in eq. (5.1) the $(\Delta\hat{X})$ and $(\Delta\hat{P})$ is refereed at the apparatus that measurement the position and the momentum of one particle. Finally, the uncertainties relation cause limitations on the measurement precision and improve the accuracy of the measurement devices. So, the study of these uncertainties it is very important.

5.1.1 Uncertainty relations and QBM models

In this section, using the Peres-Horodecki criterion and the Wigner function, we study the separability of a bipartite state. In the phase space coordinates the commutation relation, take the form:

$$[\xi_a, \xi_b] = i\Omega \tag{5.6}$$

with $a, b = 1, 2, \ldots, 2N$, and the symplectic matrix

$$\Omega = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \tag{5.7}$$

with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.8}$$

In the Wigner distribution and with the partial transpose operation, we have [59]

$$W(q_1, p_1, q_2, p_2) \longrightarrow W(q_1, p_1, q_2, -p_2) \tag{5.9}$$

which means that the mirror reflection acts only the p_2.

$$\xi \longrightarrow \Lambda\xi, \Lambda = diag(1, 1, 1, -1) \tag{5.10}$$

So, the Peres-Horodecki criterion becomes as: [59, 60]

Definition 2. *If $\hat{\rho}$ is separable, then its Wigner distribution necessarily over into a Wigner distribution under the phase space mirror reflection Λ.*

If we work for the variables:

$$\hat{X}_{\pm} = \hat{X}_1 + \pm\hat{X}_2 \tag{5.11}$$
$$\hat{P}_{\pm} = \hat{P}_1 + \pm\hat{P}_2 \tag{5.12}$$

the Peres-Horodecki partial transpose operation acts only at the momentum and we have:

$$\Lambda(\hat{X}_+, \hat{P}_+, \hat{X}_-, \hat{P}_-) = (\hat{X}_+, \hat{P}_-, \hat{X}_-, \hat{P}_-) \tag{5.13}$$

and therefore, the uncertainties become, [29]

$$A_{X_+P_+} := (\Delta X_+^2)(\Delta P_+)^2 - V_{X_+P_+}^2 \geq \frac{1}{4} \tag{5.14}$$
$$A_{X_-P_-} := (\Delta X_-^2)(\Delta P_-)^2 - V_{X_-P_-}^2 \geq \frac{1}{4} \tag{5.15}$$

and

$$A_{X_+P_-} := (\Delta X_+^2)(\Delta P_-)^2 - V_{X_+P_-}^2 \geq \frac{1}{4} \tag{5.16}$$
$$A_{X_-P_+} := (\Delta X_-^2)(\Delta P_+)^2 - V_{X_-P_+}^2 \geq \frac{1}{4} \tag{5.17}$$

The quantities $A_{X_+P_-}$,$A_{X_-P_+}$ and the inequalities eq.(5.16, 5.17) play an important role for the characterisation of any quantum state. When any quantum state satisfies the inequalities eq.(5.16, 5.17) then this state is factorized. On the other hand, if the inequalities violated, then the quantum state is entangled.

The uncertainty relations play an important role in quantum mechanics. First of all, they introduce limitations at the measurement precision, and second, they are a useful tool for the determination of precision of measurement apparatus. If we consider the covariance matrix V_t, as

$$V_{ab} = \langle \{\Delta\hat{\xi}_a, \Delta\hat{\xi}_b\} \rangle \tag{5.18}$$

we can define the standard uncertainty relation as the following form:

$$V_0 + \frac{i}{2}\tilde{\Omega} \geq 0 \tag{5.19}$$

where

$$\tilde{\Omega} = \Lambda\Omega\Lambda \tag{5.20}$$

i.e.

$$\Omega = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{5.21}$$

and

$$V_t = R(t)V_0R^T(t) + S(t) \tag{5.22}$$

where V_0: is the correlation function of initial state.

In this section, we define the matrix V_0 for two cases of initial state. The first one is the state $|\psi >= |0 > |0 >$ and the second one is the cat state with wave-function $|\psi = \frac{1}{\sqrt{2}}(|0 > |0 > \pm|0 > |0 >)$. In the first case, we have that.

$$V_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.23}$$

and in the second case

$$V_0 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \tag{5.24}$$

The correlation function of initial state satisfies the inequality (5.19), so

$$V_t \geq -\frac{i}{2}R(t)\Omega R^T(t) + S(t) \tag{5.25}$$

The inequality (5.25) is a generalized uncertainty relation that contains the effect of the environment. It depends on the Wigner function propagator and specifically on the matrices $R(t)$, $S(t)$. This inequality describes the correlation matrix at time t, and the equality is achieved only for Gaussian states [29].

5.1.2 Covariance matrix

The two-point correlation matrix V of a quantum state $\hat{\rho}$, defined by:

$$V_{ab} = \frac{1}{2}Tr[\hat{\rho}(\hat{\xi}_a\hat{\xi}b + \hat{\xi}_b\hat{\xi}_b)] - Tr(\hat{\rho}\hat{\xi}_a)Tr(\hat{\xi}_b) \tag{5.26}$$

where

$$V_t = R(t)V_0R^T(t) + S(t) \tag{5.27}$$

and

$$R = \left(\begin{array}{c|c} \dot{u}(t) & u(t)M^{-1} \\ \hline M\ddot{u}(t) & M\dot{u}(t)M^{-1} \end{array} \right), \tag{5.28}$$

$$R^T(t) = \left(\begin{array}{c|c} \dot{u}(t) & M\ddot{u}(t) \\ \hline u(t)M^{-1} & M\dot{u}(t)M^{-1} \end{array} \right) \tag{5.29}$$

V_0: is the correlation matrix of the initial state and satisfies the inequality.
Given that,

$$V \geq -\frac{i}{2}\Omega \tag{5.30}$$

we have

$$V(t) \geq -\frac{i}{2}R(t)\Omega R^T(t) + S(t). \tag{5.31}$$

Specifically, for our model, we calculate the covariance matrix analytically, and we find that this matrix has the form

$$V(t) \geq -\frac{i}{2}\begin{pmatrix} V_{X_+X_+}(t) & V_{X_+X_-}(t) & V_{X_+P_+}(t) & V_{X_+P_-}(t) \\ V_{X_-X_+}(t) & V_{X_-X_-}(t) & V_{X_-P_+}(t) & V_{X_-P_+}(t) \\ V_{P_+X_+}(t) & V_{P_+X_-}(t) & V_{P_+P_+}(t) & V_{P_+P_-}(t) \\ V_{P_-X_+}(t) & V_{P_-X_-}(t) & V_{P_-P_+}(t) & V_{P_-P_-}(t) \end{pmatrix} + S(t) \tag{5.32}$$

where

$$V_{X_+X_+}(t) = S_{X_1X_1}(t) + S_{X_2X_1}(t) \tag{5.33}$$
$$V_{X_+X_-}(t) = 0 \tag{5.34}$$
$$V_{X_-X_+}(t) = 0 \tag{5.35}$$
$$V_{X_-X_-}(t) = S_{X_1X_1}(t) - S_{X_2X_1}(t) \tag{5.36}$$

$$V(t) \geq -\frac{i}{2}\begin{pmatrix} V_{P_+P_+}(t) & V_{P_+P_-}(t) \\ V_{P_-P_+}(t) & V_{P_-P_-}(t) \end{pmatrix} \tag{5.37}$$

$$V_{P_+P_+}(t) = S_{P_1P_1}(t) + S_{_2P_1}(t) \tag{5.38}$$
$$V_{P_+P_-}(t) = 0 \tag{5.39}$$
$$V_{P_-P_+}(t) = 0 \tag{5.40}$$
$$V_{P_-P_-}(t) = S_{P_1P_1}(t) - S_{P_1P_2}(t) \tag{5.41}$$

$$V(t) \geq -\frac{i}{2}\begin{pmatrix} V_{X_+P_+}(t) & V_{X_+P_-}(t) \\ V_{X_-P_+}(t) & V_{X_-P_-}(t) \end{pmatrix} \tag{5.42}$$

$$V_{X_+P_+}(t) = E(t) + C(t) + S_{X_1P_1}(t) + S_{X_2P_1}(t) \tag{5.43}$$
$$V_{X_+P_-}(t) = 0 \tag{5.44}$$
$$V_{X_-P_+}(t) = 0 \tag{5.45}$$
$$V_{X_-P_-}(t) = E(t) - C(t) + S_{X_1P_1}(t) - S_{X_1P_2}(t) \tag{5.46}$$

where

$$E(t) = -M(\ddot{u}_{11}(t)u_{11}(t) + \ddot{u}_{12}(t)u_{12}(t)M + M(\dot{u}_{11}^2(t) + \dot{u}_{12}^2(t))M^{-1} \tag{5.47}$$
$$C(t) = -M(\ddot{u}_{11}(t)u_{12}(t) + \ddot{u}_{12}(t)u_{11}(t)M + M(\dot{u}_{12}(t) + \dot{u}_{11}(t))M^{-1} \tag{5.48}$$

Finally, we calculate the covariance matrix V_t analytically and arithmetical using the Wolfram Mathematica. This covariance matrix is necessary for the study of the factorizability of a given state at time t.

5.2 Entanglement: Theoretical background

In this section, we are going to note some applications of our model in quantum information theory and quantum computation. The first one is to understand an important fundamental quantum phenomenon, the entanglement.

5.2.1 Partial Transpose Criterion

We consider a density matrix $\hat{\rho}$ at the Hilbert space $H_1 \otimes H_2$ and an orthogonal basis $|n\rangle$ at Hilbert space H_1 and an orthogonal basis $|m\rangle$ at Hilbert space H_2. Also, we define the partial trace of $\hat{\rho}$, the density matrix $\hat{\rho}_2$ at Hilbert space H_2 as $\hat{\rho}_2 = Tr_{H_1}\hat{\rho}$. The states $\hat{\rho}_1, \hat{\rho}_2$ often called reduced density matrix and contain the information of the total system, that extract from the measure of the first or second subsystem. There is a criterion for checking if a given quantum state is entangled, that proposed by Schmidt.

Definition 3. Schmidt Decomposition *Even vector $|\psi\rangle \in H_1 \otimes H_2$ with dimension N^2 can be written as:*

$$|\psi\rangle = \sum_{i=1}^{N} c_1|\phi_1\rangle \otimes |\psi_i\rangle \tag{5.49}$$

where $c_i > 0$ and the $|\phi_i\rangle, |\psi_i\rangle$ can be called the Schmidt modes, and defines in terms of the orthogonal basis of the Hilbert space H_1, H_2, respectively. The number N is called Schmidt number of vector $|\psi\rangle$ and is smaller or equal of the dimensions of H_1 and H_2.

So, we can see that if $N = 1$, the state $|\psi\rangle$ is separable, and if $N\rangle 1$, the state is entangled. Using the Schmidt decomposition, we can write the reduced density matrix $\hat{\rho}_1$ and $\hat{\rho}_2$ as

$$\hat{\rho}_1 = \sum_1^N |c_i|^2 |\phi_i\rangle\langle\phi_i| \tag{5.50}$$

$$\hat{\rho}_2 = \sum_1^N |c_i|^2 |\psi_i\rangle\langle\psi_i| \tag{5.51}$$

The pure state $|\psi\rangle$ is entangled if it can not written as $|\psi\rangle = |\psi_i\rangle|\phi_1\rangle$.

In the particular case where $N = dimH_1 = dimH_2$, the reduced density matrices are the maximum mixed states, and therefore the state $|psi\rangle$ is the state of maximum entanglement.

In the case of mixed states $\hat{\rho} = \sum_i p_i |\phi_i\rangle\langle\phi_i|$, $\hat{\rho}$ is called separable if it can be written as $\hat{\rho} = \sum_j p_j \hat{\rho}_j^a \hat{\rho}_j^b$, $\sum_j p_j = 1$. If it is not separable, it called entangled.

Beyond of the Schmidt Decomposition, Peres and Horodecki ([59],[60]) propose a criterion, called Positive Partial Transpose (PPT) for the checking of the separability of any given bipartite continuous-variable state. According to this criterion, if any given state has an operator that can not be density matrix (i.e., it has one negative eigenvalue), then the state $\hat{\rho}$ is entangled. The PPT criterion is a necessary condition for checking the separability, but it is not sufficient. Only in the 2×2 and 2×3 dimensional cases, the PPT criterion can be a necessary and sufficient condition for separability. In our research, using the PPT, we show that the open system dynamics of the system of two detectors lead to a different asymptotic state.

5.2.2 Entanglement dynamics

Many of the most modern technologies use quantum theory and several quantum phenomena, such as superposition and entanglement, in practical applications, which perform better than similar technologies based on classical physics theory [31]. Most of these applications are mainly related to quantum computers, quantum cryptography, and quantum simulation [32]. For the case of Gaussian states, have been developed a variety of the measures of the entanglement, such as the negativity E_N. This measure can be calculated in the Wigner function distribution and using the symplectic eigenvalues of the covariance matrix, V_t. If the covariance matrix has the form:

$$V_t = \begin{pmatrix} A & B \\ M & M^T \end{pmatrix} \tag{5.52}$$

where

$$A = \begin{pmatrix} \langle X_1^2 \rangle & \left\langle \frac{X_1 P_1 + P_1 X_1}{2} \right\rangle \\ \left\langle \frac{X_1 P_1 + P_1 X_1}{2} \right\rangle & \langle P_1^2 \rangle \end{pmatrix} \tag{5.53}$$

$$B = \begin{pmatrix} \langle X_2^2 \rangle & \left\langle \frac{X_2P_2+P_2X_2}{2} \right\rangle \\ \left\langle \frac{X_2P_2+P_2X_2}{2} \right\rangle & \langle P_2^2 \rangle \end{pmatrix} \tag{5.54}$$

and

$$M = \begin{pmatrix} \left\langle \frac{X_1X_2+X_2X_1}{2} \right\rangle & \left\langle \frac{X_1P_2+P_2X_1}{2} \right\rangle \\ \left\langle \frac{X_2P_1+P_1X_2}{2} \right\rangle & \left\langle \frac{P_1P_2+P_2P_1}{2} \right\rangle \end{pmatrix} \tag{5.55}$$

According to the Positive Partial Transpose criterion by Peres-Horodecki, the minimum eigenvalue of the transpose of V_t must satisfy the equality:

$$\lambda_{\pm}^2 = \frac{\left(\Delta(V_t) \pm \sqrt{\Delta(V_t)^2 - 4Det(V_t)}\right)}{2} \tag{5.56}$$

where

$$\Delta(V_t) = Det(A) + Det(B) - 2Det(M) \tag{5.57}$$

When $\lambda_{min} \geq \frac{1}{2}$, the Gaussian states are separable, and for the measurement of entanglement, we can use the logarithmic negativity. This measure is given by the following equation:

$$E_N = max\{0, -ln(2l_{min})\} \tag{5.58}$$

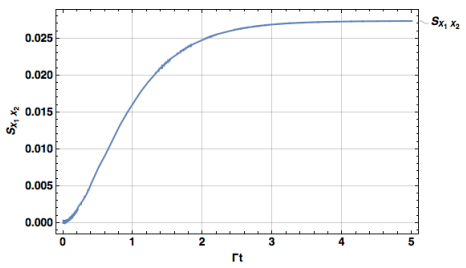
Using the PPT criterion, we investigate the dependence of entanglement on the distance r. First, we study the case where the initial state is a vacuum state $|\psi(t)\rangle = |0\rangle|0\rangle$. For this state, we calculate the correlation matrix of the initial state, and using this, we continue with the definition of the covariance matrix V_t.

5.3 Asymptotic states and generation of entanglement

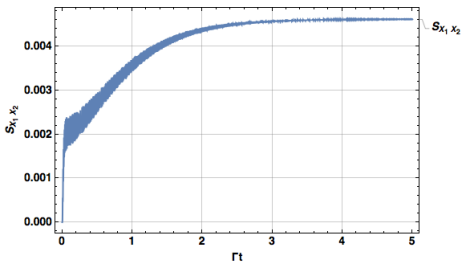
In this section, using the PPT criterion, we show that the open system dynamics of our model, i.e., the detectors, lead to a unique asymptotic state. This state is correlated, and it is entangled for small separations.

5.3.1 Asymptotic state

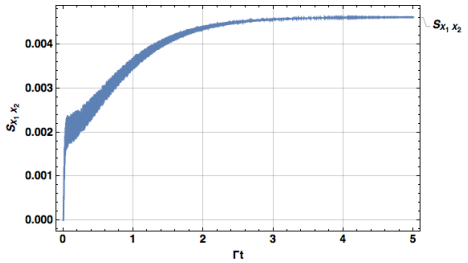
In Chap. 2, we showed that the reduced density matrix propagator for this model is fully determined by the matrices $R(t)$ and $S(t)$. In Sec. 4.2, we



(A) $S_{X_1X_2}$, $\Gamma_0 r = 0.01$



(B) $S_{X_1X_2}$, $\Gamma_0 r = 0.1$



(C) $S_{X_1X_2}$, $\Gamma_0 r = 1$.

FIGURE 5.1: Evolution of the correlation functions $S_{X_1X_2}$, as a function of Γt and for different values of $\Gamma_0 r$. In this plot $\Gamma_0/\Omega = 10^{-3}$.

evaluated $R(t)$ and showed its non-Markovian behaviour for $\Gamma r \geq 1$. The matrix $S(t)$ is determined by Eqs. (4.15—4.17).

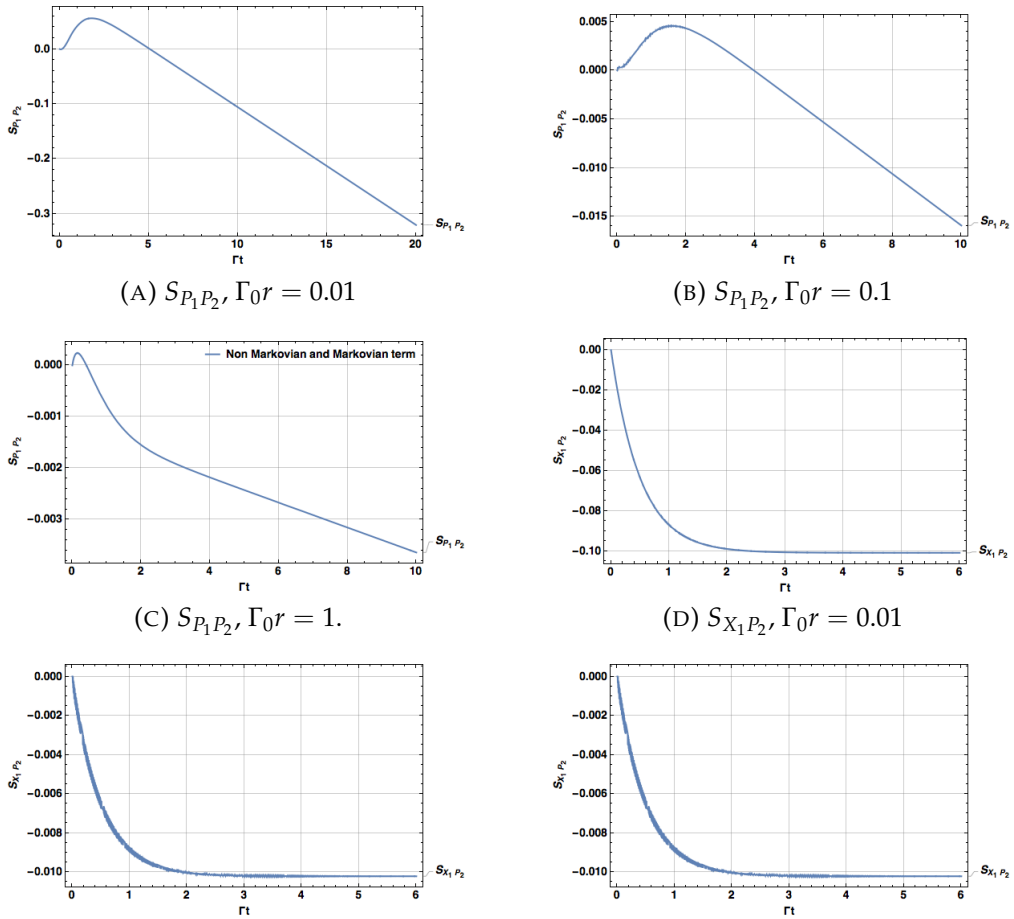
When evaluating the matrix elements $S_{ab}(t)$, we find that even for the non-diagonal elements the dominant contribution comes from the functions $u_{11}(t)$ and $u_{22}(t)$ and their derivatives. These functions are well described by the pole term except for very long times. Hence, we expect that the WWA is accurate for $S_{ab}(t)$. Numerically, we find that the difference between S_{ab} calculated via the WWA and the exact expression is of the order of $\Gamma_0/\Omega << 1$. If we substitute the pole term for $u(t)$ in Eqs solely. (4.15—4.17), integration can be carried out analytically. They lead to an analytic expression for $S_{ab}(t)$ that is accurate to order Γ_0/Ω.

The functions $u_{\alpha\alpha'}(t)$ vanish as $t \to \infty$, hence, so does the matrix $R_{ab}(t)$. Eq. (4.7) implies that as $t \to 0$, the Wigner function propagator becomes independent of ξ_0. Numerical evaluation of $S_{ab}(t)$ shows that it asymptotic to a constant matrix for large t—we denote this matrix by $S(\infty)$. Hence, asymptotically the system is described by the Wigner function

$$W_\infty(\xi) = \frac{1}{\pi\sqrt{\det S(\infty)}} \exp\left[-\frac{1}{2}S^{-1}_{ab}(\infty)\xi^a\xi^b\right], \tag{5.59}$$

By Eq. (4.10), the correlation matrix at infinity $V_{ab}(\infty)$ coincides with $S_{ab}(\infty)$.

Interestingly, the matrix $S(\infty)$ involves correlations between the two detectors: the matrix elements $S_{X_1X_2}(\infty)$, $S_{P_1P_2}(\infty)$ and $S_{X_1P_2}(\infty)$ that describe such correlations are non-zero. To see this, we use the fact that the dominant contribution to $S_{ab}(\infty)$ is well approximated by the WWA. Substituting Eq. (4.39) into Eqs. (4.15—4.17), taking the limit $t \to \infty$, and keeping terms to



(A) $S_{P_1P_2}$, $\Gamma_0 r = 0.01$ (B) $S_{P_1P_2}$, $\Gamma_0 r = 0.1$

(C) $S_{P_1P_2}$, $\Gamma_0 r = 1.$ (D) $S_{X_1P_2}$, $\Gamma_0 r = 0.01$

(E) $S_{X_1P_2}$, $\Gamma r = 0.1$ (F) $S_{X_1P_2}$, $\Gamma r = 1.$

FIGURE 5.2: Evolution of the correlation functions $S_{P_1P_2}$, $S_{X_1P_2}$ as a function of Γt and for different values of $\Gamma_0 r$. In this plot $\Gamma_0/\Omega = 10^{-3}$.

leading order in Γ_0/Ω, we obtain

$$S_{X_1X_1}(\infty) = S_{X_2X_2}(\infty) = \frac{\Gamma_0}{\Omega}\left[\frac{1}{\Gamma_+} + \frac{1}{\Gamma_-} - \frac{1}{2\Omega r}\left(\frac{\sin(\Omega_+ r)}{\Gamma_+} - \frac{\sin(\Omega_- r)}{\Gamma_-}\right)\right] \tag{5.60}$$

$$S_{P_1P_1}(\infty) = S_{P_2P_2}(\infty) = \Gamma_0\Omega\left[\frac{1}{\Gamma_+} + \frac{1}{\Gamma_-} - \frac{1}{2\Omega r}\left(\frac{\sin(\Omega_+ r)}{\Gamma_+} - \frac{\sin(\Omega_- r)}{\Gamma_-}\right)\right] \tag{5.61}$$

$$S_{X_1P_1}(\infty) = S_{X_2P_2}(\infty) = \frac{2\Gamma_0}{\Omega}\left(\frac{\delta\Omega}{\Omega} + \frac{\sin(\Omega_+ r) - \sin(\Omega_- r)}{4\Omega r}\right) \tag{5.62}$$

$$\tag{5.63}$$

$$S_{X_1X_2}(\infty) = S_{X_2X_1}(\infty) = \frac{\Gamma_0}{\Omega}\left[\frac{1}{\Gamma_+} - \frac{1}{\Gamma_-} - \frac{1}{2\Omega r}\left(\frac{\sin(\Omega_+ r)}{\Gamma_+} + \frac{\sin(\Omega_- r)}{\Gamma_-}\right)\right] \tag{5.64}$$

$$S_{P_1P_2}(\infty) = S_{P_2P_1}(\infty) = \Gamma_0\Omega\left[\frac{1}{\Gamma_+} - \frac{1}{\Gamma_-} - \frac{1}{2\Omega r}\left(\frac{\sin(\Omega_+ r)}{\Gamma_+} + \frac{\sin(\Omega_- r)}{\Gamma_-}\right)\right] \tag{5.65}$$

$$S_{X_1P_2}(\infty) = S_{X_2P_1}(\infty) = \frac{\Gamma_0}{\Omega}\left(-1 + \frac{\sin(\Omega_+ r) + \sin(\Omega_- r)}{2\Omega r}\right) \tag{5.66}$$

Remarkably, the correlation terms $S_{X_1X_2}$ and $S_{P_1P_2}$ turn out to be of order $(\Gamma_0/\Omega)^0$, i.e., of the same order with the diagonal terms. However, unlike the diagonal terms, correlation terms are suppressed as Ωr becomes significantly larger than unity. For $\Omega r \simeq 20$ or smaller, there is significant residual correlation between the detectors. This may appear surprising, but we note that the destruction of correlations is a common feature of either high-temperature baths, or systems of qubits, and not a generic property of open quantum systems. The existence of asymptotic correlations appears more intuitive when viewing the oscillators as actual particle detectors. We would expect the detectors to develop correlations if they dominantly interact with particles with de Broglie wavelength of the order of their distance[1].

Next, we examine whether the asymptotic state is entangled. To this end, we employ the Positive Partial Transpose (PPT) separability criterion of Peres and Horodecki [59, 60]. In the present context, the PPT criterion is applied to the correlation matrix V. A correlation matrix on $L^2(\boldsymbol{R}) \otimes L^2(\boldsymbol{R})$ is separable if it satisfies

$$V \geq -\frac{i}{2}\tilde{\Omega}, \quad \tilde{\Omega} = \Lambda\Omega\Lambda \tag{5.67}$$

where Ω is the symplectic form on the four-dimensional phase space of two particles and Λ is the matrix of the PPT operation $\Lambda = \mathrm{diag}(1, 1, 1, -1)$ [61].

In Fig.(5.4), we plot the minimal eigenvalue of $S(\infty) + \frac{i}{2}\tilde{\Omega}$ as a function of Ωr. A negative value of λ_- indicates an entangled Gaussian state, a positive value of a separable Gaussian state. We see that the asymptotic state is entangled for $\Omega r \lessapprox 1.79$ and that the entanglement is stronger as $r \to 0$. The results are qualitatively compatible with the analysis of Ref. [18] (that ignores backreaction) and the analysis of Ref. [58] (that employs a perturbation expansion scheme). We note that Eqs. (5.60—5.66) provide the exact asymptotic expression of S in the weak coupling limit.

5.3.2 Entanglement generation

Having established the asymptotic behavior of the two-detector system, and identified the asymptotic behavior of entanglement, we examine how entanglement is generated in time. Again, we employ the separability criterion (5.67). We consider an initial factorized state $|z\rangle \otimes |z'\rangle$, which is a product of coherent states. In Fig. (5.5), we plot the lowest eigenvalue of $V_t + \frac{i}{2}\tilde{\Omega}$ as a function of $\Gamma_0 t$, where V_t is given by Eq. (4.10). As expected, entanglement is generated only at early times.

[1] There is no lower limit to Ω in our model—except for the infra-red cut-off— so the detectors could be correlated even if they are separated by microscopically large distances. Of course, actual particle detectors are macroscopic systems, and the variables $\hat{X}_\alpha$ are highly coarse-grained. The inclusion of additional degrees of freedom to the detector would introduce decoherence effects that would suppress such correlations beyond some length scale L.

The choice of the initial state $|z\rangle \otimes |z'\rangle$ does not significantly affect the entanglement creation. Other factorized initial states exhibit the same behavior.

For $z = z' = 0$, the initial state is $|0,0\rangle$, i.e., the ground state of the system of two oscillators. However, this state is not the lowest energy state for the full field-detector Hamiltonian. For this reason, the energy of the detector degrees of freedom momentarily increases as a result of the interaction with the environment, which would be paradoxical if $|0,0\rangle$ were a true ground state.

The state $|0,0\rangle$ may be viewed as a ground state of the system if we can assume a set-up in which the field-detector coupling switches on at $t = 0$. As long as the switching on takes place at time-scales much smaller than Γ^{-1}, the solutions to the reduced dynamics derived here are applicable.

In this context, the creation of entanglement from an initial vacuum state is referred to as *harvesting* of the QFT vacuum. Most research on harvesting focuses on the evaluation of the effect in the lowest order of time-dependent perturbation theory. This is a good approximation as long as the interaction is switched on for a time interval much smaller than the relaxation time. For longer times, an open-quantum system treatment that takes backreaction into account is essential. To see this, note that for small separations between the two detectors, entanglement is generated at early times, but this entanglement is degraded in time, leading to an asymptotic state with classical correlations. This implies that studies of entanglement extraction that ignore backreaction may significantly overestimate the amount of harvested entanglement.

Finally, we note that there is no significant generation of entanglement outside the light-cone for static detectors.

5.4 The challenge of causality

An important motivation of this work is to understand how causality is implemented in the communication of separated localized quantum objects that are interacting through a massless quantum field. The present model, being exactly solvable, provides an explicit demonstration of Fermi's two-atom problem, in which the fundamental physical issues are not obscured by questions about the validity of approximations.

It is straightforward to verify that the classical equations of motion (4.13) are not causal: $X_2(t)$ depends on the value of $X_1(0)$, even for times $t < r$. In some sense, this result is to be expected. Eq. (4.13) describes the interaction between the oscillators in terms of direct coupling in position—even if it is non-local in time—and it is well known that direct particle coupling cannot lead to causal dynamics in relativistic systems. The problem is that Eq. (4.13) describes the evolution of the expectation values of the observables $\hat{X}_{1,2}$; hence, its non-causal behavior seemingly implies superluminal signals.

Having an exactly solvable model allows us to demonstrate explicitly that this non-causal behavior is not an artifact of common approximations employed in such systems—for causality violation in Unruh-DeWitt detectors in a perturbative evaluation, see [62, 63]. First, non-causality is not due to the

choice of a factorizing initial condition, that was employed in the derivation of the density matrix propagator. Such a condition is arguably unphysical because any preparation of the system cannot affect arbitrarily high energies of the field. Factorizability holds at most up to a cut-off energy scale. However, as mentioned in Sec. (4)existing models in the theory of open quantum systems strongly suggest such correlations are mostly significant at early times and that their effects become negligible as correlations are established between system and environment due to dynamical interaction.

More importantly, we can derive an exact evolution equation for the expectation value $\langle \hat{X}_r \rangle$ [29]

$$\frac{d^2}{dt^2}\langle \hat{X}_\alpha(t)\rangle + \Omega_\alpha^2 \langle \hat{X}_\alpha(t)\rangle + 2\sum_{\alpha'}\int_0^t \gamma_{\alpha\alpha'}(t-s)\langle \hat{X}_{\alpha'}(s)\rangle = \sum_i \frac{c_{i\alpha}}{M_r}\langle \hat{q}_i(t)\rangle, \quad (5.68)$$

where $\hat{q}_i$ is the field operator associated to the i-th mode, evolving according to the free equations of motion for the field. We can also choose the initial state to satisfy $\langle \phi(x)\rangle = \langle \pi(x)\rangle = 0$, where $\hat{\pi}(x)$ is the field conjugate momentum[2]. This condition implies that $\langle \hat{q}_i(t)\rangle = 0$, hence, $\langle \hat{X}_\alpha(t)\rangle$ satisfies to Eq. (4.13). Mean values evolve non-causally, irrespective of the initial condition.

The situation is analogous to that of Fermi's two-level atom that was mentioned in the introduction. In this sense, it is generic to all relativistic systems with well-localized subsystems. Hegerfeld proved with minimal assumptions that for any systems A and B, in disjoint regions, that interacting through a quantum field, the excitation probability of B is nonzero immediately after $t = 0$ [15]. The present model exemplifies Hegerfeld's theorem in an exactly solvable system.

Hence, this type of non-causality is not a feature of unphysical dynamics, for example, due to the limited validity of the field-particle coupling of this model. To see this, note that field-particle couplings can be derived for the dynamics of a N-level atom coupled to the electromagnetic field [64]. The harmonic oscillators considered here can be viewed as atoms with equal spacing in the levels and $N \to \infty$. The starting point in such derivations is the full Quantum Electrodynamics. The crucial condition that leads to couplings of the form (4.18) is the *dipole approximation*. This asserts that the size of the localized systems is much smaller than the wavelength of the emitted radiation. Since the size of those systems defines the cut-off frequency of Λ, the dipole approximation is expected to hold with an accuracy of the order of Ω/Λ. Hence, corrections to the dipole approximation (and, hence, to the field particle coupling) are expected to increase with Ω and to be sensitive on the cut-off Λ. This is the case for the runaway solutions that are regularized away—see Sec. 3.2. In contrast, the non-causal behavior that characterizes Eq. (4.13) is insensitive to Ω or Λ.

[2]This is a natural condition for a state that behaves like the field vacuum. In any case, the mean value of the field and its conjugate momentum can be shifted to any value by a unitary action of the Weyl group, that is generated by the field canonical algebra.

For this reason, we believe that the problem of causality in detector-field interactions is fundamentally *kinematical* and not dynamical. This is supported by several theorems on the impossibility to define localization observables in relativistic quantum systems [65, 66, 67]. Existing definitions of localized observables conflict the requirement of relativistic causality. Observables that appear to be local and causal in classical theory or non-relativistic quantum theory (e.g., a particle's position) fail to be so in relativistic quantum theory. In particular, this is the case for the quantities $\hat{X}_\alpha$ and $\hat{P}_\alpha$ that describe the degrees of freedom of the oscillator detectors in the present model. Once the interaction with the field is present, they canno longer be viewed as localized observables pertaining to a single detector. Being non-local observables, their non-causal evolution is not problematic.

This also means that a causal description of the relativistic transmission of information requires a consistent definition of localized observables. The Hilbert space of the system $\mathcal{H}_{tot} = \mathcal{H}_{d1} \otimes \mathcal{H}_{d2} \otimes \mathcal{H}_{field}$, where $\mathcal{H}_{d\alpha}$ is a Hilbert space associated to the α detector and $\mathcal{H}_{field}$ the field Hilbert space. An operator that corresponds to a measurement in the detector 1 should not be of the form $\hat{A} \otimes \hat{I} \otimes \hat{I}$. Still, rather, it should be a non-factorized operator on $\mathcal{H}_{tot}$ that reduces to the factorizing form for $\lambda \to 0$. Rather heuristically, a local observable should include a contribution virtual photons before in order to be compatible with causality [16].

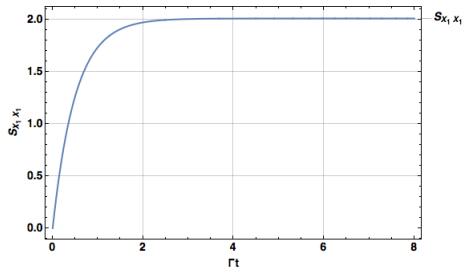
It is doubtful that self-adjoint operators that generalize $\hat{X}_\alpha$ and $\hat{P}_\alpha$ for the interacting system can be defined in a way that is compatible with causality. There are strong arguments-that ideal measurements—i.e., measurements corresponding to self-adjoint operators—are incompatible with causality in QFT [68]. These arguments are completely independent of Fermi's two-atom problem. This means that we must express QFT measurements in terms of Positive-Operator-Valued measures (POVMs). One of us has proposed the use of time-extended observables for the description of particle localization [69]. Time extended observables correspond to POVMs that partly depend upon the dynamics of the quantum system [70]. Hence, a model with exactly solvable dynamics, such as the one analyzed here, is important for the explicit construction of such observables and for testing their causal behavior.

Implications to entanglement generation. We argued that operators of the form $\hat{A} \otimes \hat{I} \otimes \hat{I}$ could not be viewed as corresponding to a local measurement of the first detector, and similarly for operators of the form $\hat{I} \otimes \hat{A} \otimes \hat{I}$ in relation to the second detector. However, the representation of local measurements with operators of this form is a cornerstone of quantum information theory. In particular, it is a prerequisite for identifying entanglement as a quantum resource. Of course, this representation is based fundamentally on non-relativistic quantum physics. It does not directly apply to relativistic quantum systems, and it does not incorporate the severe restrictions raised by QFT.

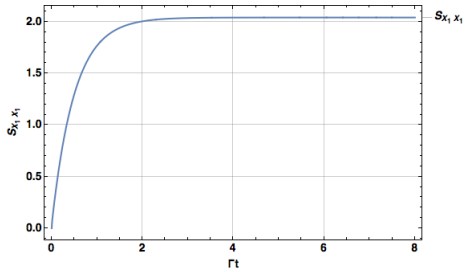
Hence, there is no fundamental justification that the usual measures of entanglement between the detectors define a genuine quantum resource when the detectors are coupled to a quantum field. One cannot assert that these measures describe non-classical correlations between localized measurements.

Hence, the physical relevance of entanglement harvested by the vacuum of entanglement generation outside the light-cone questionable.

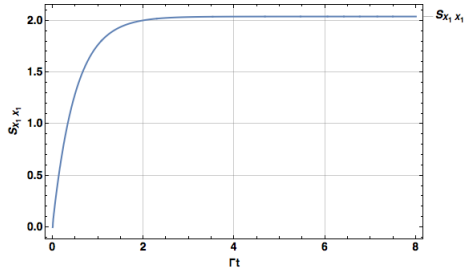
In our opinion, we must first resolve the issue of defining an appropriate notion of localized observables in relativistic QFT, as exemplified by Fermi's problem. Then we can attempt to define a new quantum resource that represents Bell-type correlations in the QFT context.



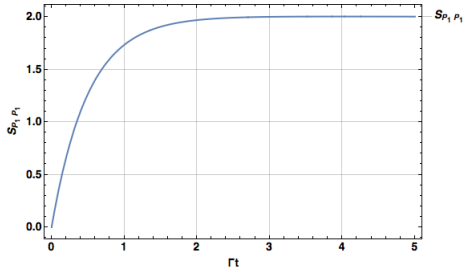
(A) $S_{X_1X_1}, \Gamma_0 r = 0.01$



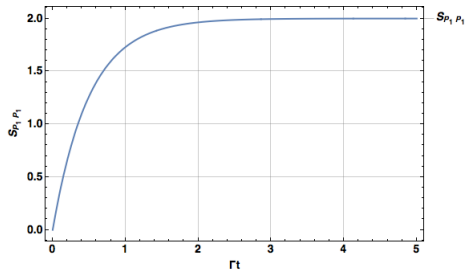
(B) $S_{X_1X_1}, \Gamma_0 r = 0.1$



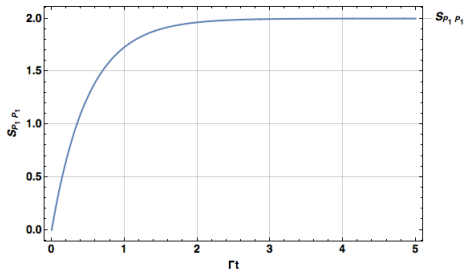
(C) $S_{X_1X_1}, \Gamma_0 r = 1.$



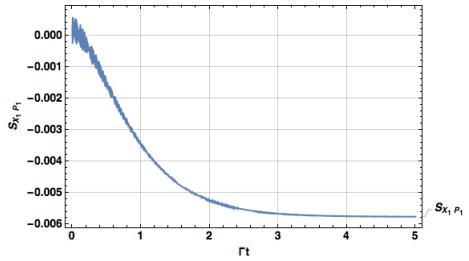
(D) $S_{P_1P_1}, \Gamma_0 r = 0.01$



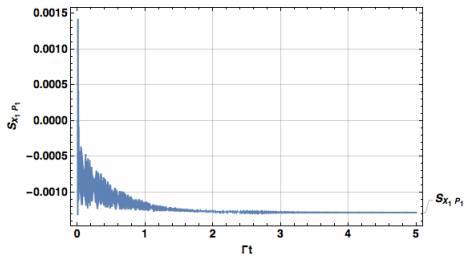
(E) $S_{P_1P_1}, \Gamma_0 r = 0.1$



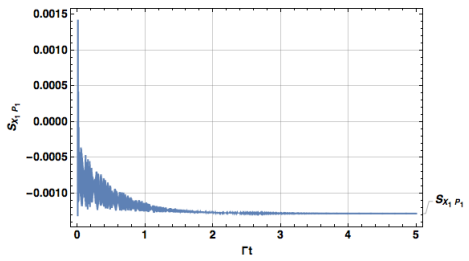
(F) $S_{P_1P_1}, \Gamma_0 r = 1.$



(G) $S_{X_1P_1}, \Gamma_0 r = 0.01$



(H) $S_{X_1P_1}, \Gamma r = 0.1$



(I) $S_{X_1P_1}, \Gamma r = 1.$

FIGURE 5.3: Evolution of the correlation functions $S_{X_1X_1}$, $S_{P_1P_1}$, $S_{X_1P_1}$ as a function of Γt and for different values of $\Gamma_0 r$. In this plot $\Gamma_0/\Omega = 10^{-3}$.

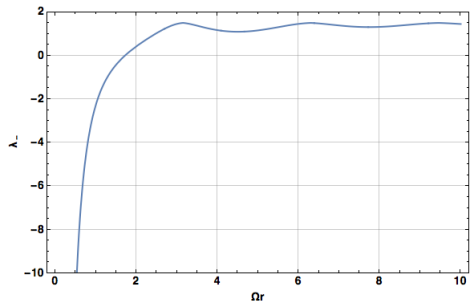
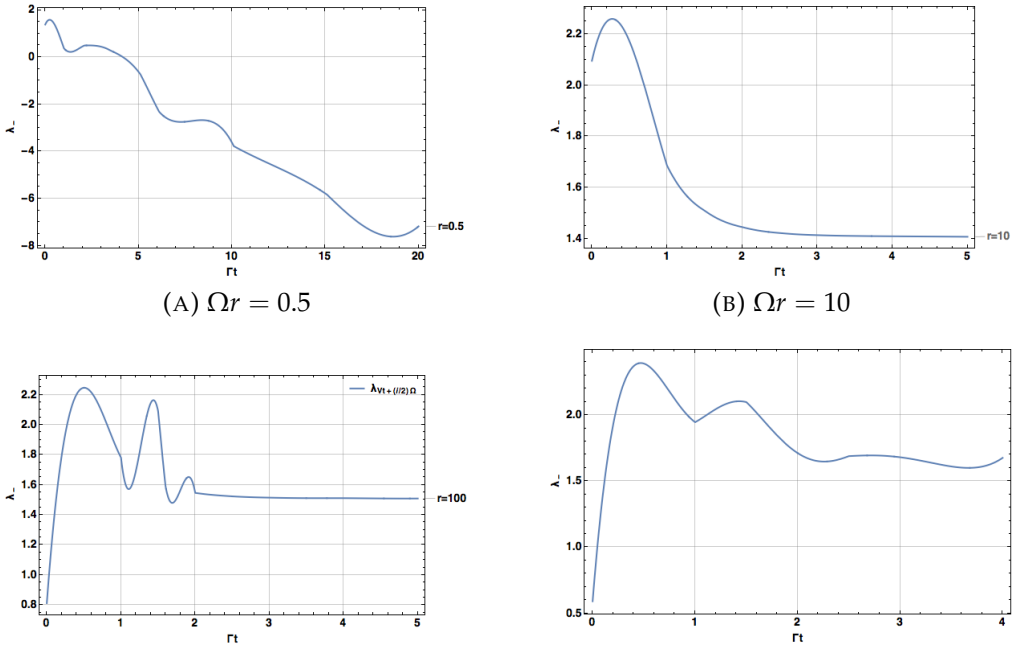


FIGURE 5.4: The minimal eigenvalue λ_- of the matrix $S(\infty) + \frac{i}{2}\tilde{\Omega}$ as a function of Ωr.



(A) $\Omega r = 0.5$

(B) $\Omega r = 10$

(C) $\Omega r = 100$

(D) $\Omega r = 1000$

FIGURE 5.5: The evolution of minimal eigenvalue λ_- of $V_t + \frac{i}{2}\tilde{\Omega}$ for initial factorized state $|z\rangle \otimes |z'\rangle$ and for different values of Ωr. We see that entanglement is generated only for small r.

5.5 Summary and Conclusion

The next-generation quantum experiments in space lie in the regime where the relativistic effects become significant. For this reason, it is crucial to investigate how separated quantum systems interact via relativistic quantum fields. Summarizing, we studied the evolution of the system of two localized detectors (oscillators) interacting through a massless quantum field in a vacuum state via an Unruh-DeWitt coupling. This system admits an exact solution is providing a good model for addressing fundamental issues in particle-field interactions, causality, and locality in quantum field measurements that are relevant to proposed quantum experiments in space. Our analysis of the exact solution leads to the following results.

1. Common approximations (Markov approximation, perturbative master equation, Wigner-Weisskopf approximation) used in the study of analogous open quantum systems fail when the distance between the detectors becomes of the order of the relaxation time or larger, i.e. $r_{det} \approx \tau_{relax}$. Even if this result is derived in a specific model, we believe that can be generalize for any system interact with the environment with Hamiltonian of the form:

$$\hat{H}_{int} = \sum_{i,r} c_{i,r} \hat{X}_r \hat{q}_i \tag{5.69}$$

   In particular, the creation of correlations between remote detectors is not well described by ordinary perturbation theory and the Markov approximation.

2. There is a unique asymptotic state that is correlated; it is not entangled unless the detector separation is of the order of magnitude of the wavelength of the exchanged quanta. So, the entanglement generation in this regime, persists at times of the order of relaxation scale, beyond the most recent studies of the entanglement generation from vacuum (harvesting) [71].

3. The evolution of seemingly localized observables is non-causal. The latter is a manifestation of Fermi's two-atom problem, albeit in an exactly solvable system. We argue that the problem of causality requires a re-examination of the notion of entanglement in relativistic systems, in particular, the physical relevance of its extraction from the quantum vacuum.

We believe that the model presented here provides an important tool for addressing foundational issues in QFT, because it has a formal exact solution, and provides full mathematical control to all approximation schemes. It may be used for constructing localized observables to address the Fermi problem, for understanding causal propagation of information in QFT, and for generalizing existing quantum information concepts to relativistic systems.

5.6 Thesis Features and Conventions

The description of the causal propagation of information between two separated harmonic oscillators, presented in this work is suited for studying information transfer and non-Markovian dynamics for any system with Hamiltonian that can be written in terms of

$$H_{int} = \sum_{i,r} \hat{X}_r \hat{q}_i. \tag{5.70}$$

Our results can be applied for any system of atoms that interacts with an electromagnetic field. Also, our exact solvable system can be applied for the studying of the Non-Markovian dynamics for a system of two-qubits in general bath and for the studying of the entanglement dynamics in two-qubits system. Finally, our exact solvable system provides an exact mathematical control for all systems are based on approximations. It may be used for the understanding of causal propagation of information in Quantum Information Field (Q.F.T.). It is also useful for the understanding of causality and locality in relativistic quantum information theory. Last but not least, it may be used for the checking of the Fermi problem at the upcoming experiments in space that can be studied at the Lunar Gateway (NASA)—see. fig. (5.6).

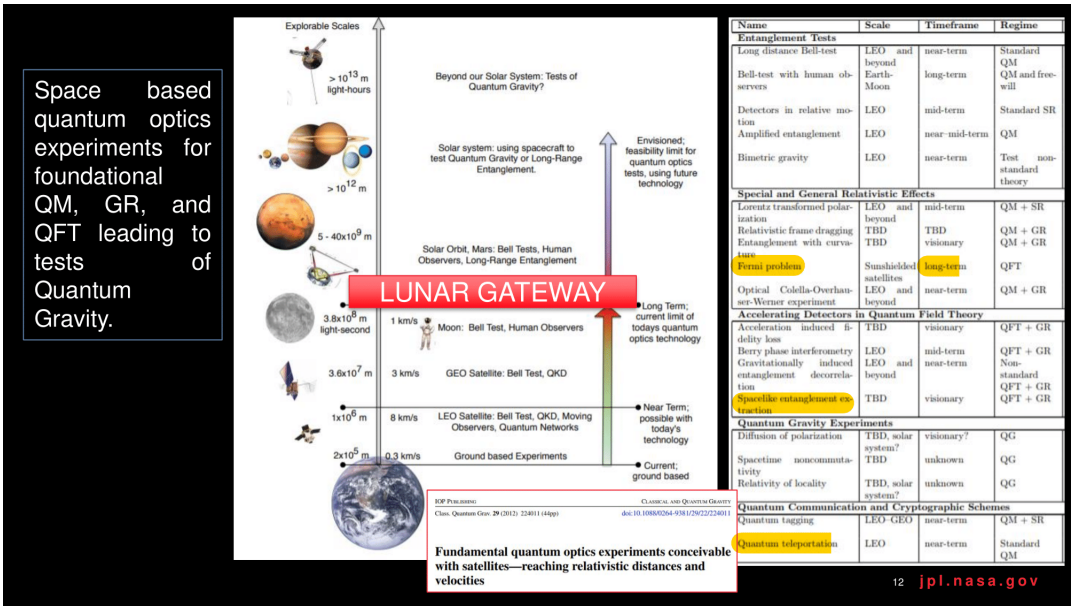


FIGURE 5.6: Representation of space based quantum optics experiments for foundational QM, GR and QFT leading to test of Quantum Gravity. (https://www.nasa.gov)

Many different applications have been left for the future. We want to continue our research by studying the case of one harmonic oscillator for the case of the strong coupling between system and environment. In appendix D, we present some of the calculations of significant quantities for the case of a system of one harmonic oscillator interacting with the environment. Also,

we would like to describe the causal propagation of information between two separated harmonic oscillators in one dimension.

Finally, for our system, we would like to determine:

1. the discussion term of a Master equation $D^{ab}(t)$, that incorporates the effect of environment-induced fluctuations,

2. and the uncertainty function $A_{X_i P_j}$, which provide the witness of entanglement of any quantum state.

Appendix A

System of two harmonic oscillator: Calculation of solution of homogeneous equation of motion $u_{rr'}(t)$

In this appendix, we give the explicit form of the solution of homogeneous part of equation of motion for the system of two harmonic oscillators.

A.1 Evaluation of the Laplace Transform Integral of matrix A^{-1}

For the case of the system of two harmonic oscillators interacting with a scalar field, in order to calculate the solution $u_{rr'}(t) = L^{-1}\tilde{A}^{-1}(z)$ of the homogeneous part of equation of motion and determine the matrix $R(t)$, it is important to calculate the matrix $A^{-1}(z)$. The matrix $\tilde{A}^{-1}(z)$ can be written as:

$$\begin{aligned}\tilde{A}^{-1}(z) &= \frac{1}{2}\left[\frac{1}{z^2+\Omega^2+2\tilde{\gamma}_0(z)+2\tilde{\gamma}_r(z)}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right.\\ &+ \left.\frac{1}{z^2+\Omega^2+2\tilde{\gamma}_{11}(z)-2\tilde{\gamma}_{12}(z)}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right], \qquad \text{(A.1)}\end{aligned}$$

so that

$$A^{-1}(t) = \frac{1}{2}\left[f_+(t)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + f_-(t)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right], \qquad \text{(A.2)}$$

in terms of

$$f_\pm(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz \frac{e^{zt}}{z^2+\Omega^2+2\tilde{\gamma}_0(z)\pm 2\tilde{\gamma}_r(z)} \qquad \text{(A.3)}$$

The integrad A.3 have a branch cut at $z = 0$ and two branches cut at $z = -\epsilon \pm i\Lambda$. So, for the calculation of the inverse Laplace, we integrate the function

$$\frac{e^{zt}}{z^2 + \Omega^2 + 2\tilde{\gamma}_0(z) \pm 2\tilde{\gamma}_r(z)} \tag{A.4}$$

using the Bromwich contour, see the fig. (A.1).

Using the Cauchy's theorem we find the functions $f_\pm(t)$ and we can see that this function is a sum of two terms:

$$f_\pm(t) = f_\pm^0 + I_\pm(t) \tag{A.5}$$

The first term of the function A.5 called *pole term and the second called branch-cut term.*

A.2 Calculation of the pole term

The calculation of poles of the eq (A.1) is done by solving the following equation:

$$z^2 + \Omega^2 + 2\tilde{\gamma}_0(z) + 2\tilde{\gamma}_r(z) = 0. \tag{A.6}$$

Using the perturbation theory for:

$$z_-^+ = -i\Omega + \lambda^2 z_1$$
$$z_+^+ = +i\Omega + \lambda^2 z_2$$

and leading order in λ^2, we have:

$$z_+^+ = -\Gamma_+ + i\Omega_+, \tag{A.7}$$
$$z_-^+ = -\Gamma_+ - i\Omega_- \tag{A.8}$$

where

$$\begin{aligned}
\Gamma_+ &= \frac{\lambda^2}{16\pi\Omega}\left(1 + \frac{\sin(r\Omega)}{r\Omega}\right), \\
\Omega_+ &= -\frac{\lambda^2}{8\pi^2\Omega}\left(\frac{\cos(r\Omega)}{r\Omega} Si(r\Omega) + ln\left(1 + \frac{\Lambda^2}{(\Omega+\epsilon)^2}\right)\right. \\
&- \left.\frac{Sin(r\Omega)}{r\Omega} Ci(r\Omega)\right) + \Omega
\end{aligned} \tag{A.9}$$

The second equation that we have to solve, is

$$z^2 + \Omega^2 + 2\tilde{\gamma}_0(z) - 2\tilde{\gamma}_r(z) = 0, \tag{A.10}$$

with:

$$\begin{aligned} z_-^- &= -i\Omega + \lambda^2 z_1, \\ z_+^- &= i\Omega + \lambda^2 z_2 \end{aligned}$$

and we have:

$$z_+^- = -\Gamma_- + i\Omega_- \tag{A.11}$$
$$z_-^- = -\Gamma_- - i\Omega_- \tag{A.12}$$

where

$$\begin{aligned} \Gamma_- &= \frac{\lambda^2}{16\pi\Omega}\left(1 - \frac{\sin(r\Omega)}{r\Omega}\right) \\ \Omega_- &= -\frac{\lambda^2}{8\pi^2\Omega}\left(-\frac{\cos(r\Omega)}{r\Omega}Si(r\Omega) + ln\left(1 + \frac{\Lambda^2}{(\Omega+\epsilon)^2}\right)\right. \\ &+ \left.\frac{Sin(r\Omega)}{r\Omega}Ci(r\Omega)\right) + \Omega \end{aligned} \tag{A.13}$$

and the $\Omega_\pm$ is the Lamb shift.

A.3 Calculation of Inverse Laplace Transform

Closing the left yields (schematically), the integrals $\int_{C_{R_A}}, \int_{C_{R_B}}, \int_{C_{R_C}}, \int_{C_{R_D}}, \int_{C_{R_\epsilon}}, \int_{C_{R_{\epsilon'}}}$, vanish as $R \to \infty$. Therefore the integral f(t) take the form:

$$f_\pm(t) = \int_{c-\imath\infty}^{c+\imath\infty} dz \frac{e^{zt}}{z^2 + \Omega^2 + 2\gamma_0(z) \pm 2\gamma_r(z)} \tag{A.14}$$
$$f_+(t) = \int_{-\infty}^{0} dz \frac{e^{zt}}{z^2 + \Omega^2 + 2\gamma_0(z) + 2\gamma_r(z)} + \int_{0}^{-\infty} dz \frac{e^{zt}}{z^2 + \Omega^2 + 2\gamma_0(z) + 2\gamma_r(z)}$$
$$f_-(t) = \int_{-\infty}^{0} dz \frac{e^{zt}}{z^2 + \Omega^2 - 2\gamma_0(z) + 2\gamma_r(z)} + \int_{0}^{-\infty} dz \frac{e^{zt}}{z^2 + \Omega^2 - 2\gamma_0(z) + 2\gamma_r(z)}$$

for $z^+ = -s + \imath\epsilon$ and $z^- = -s + \imath\epsilon$ the equation A.15 can be written as

$$\begin{aligned} I_+(t) &= -2\int_{-\infty}^{0} ds e^{-st} \\ &\left(\frac{(\gamma_0(-s^-) - \gamma_0(-s^+)) + (\gamma_r(-s^-) - \gamma_r(-s^+))}{(s^2 + \Omega^2 + 2\gamma_0(-s^+) + 2\gamma_r(-s^+))(s^2 + \Omega^2 + 2\gamma_0(-s^-) + 2\gamma_r(-s^-))}\right) \end{aligned}$$

and

$$\begin{aligned} I_-(t) &= -2\int_{-\infty}^{0} ds e^{-st} \\ &\left(\frac{(\gamma_0(-s^-) - \gamma_0(-s^+)) - (\gamma_r(-s^-) - \gamma_r(-s^+))}{(s^2 + \Omega^2 + 2\gamma_0(-s^+) - 2\gamma_r(-s^+))(s^2 + \Omega^2 + 2\gamma_0(-s^-) - 2\gamma_r(-s^-))}\right) \end{aligned}$$

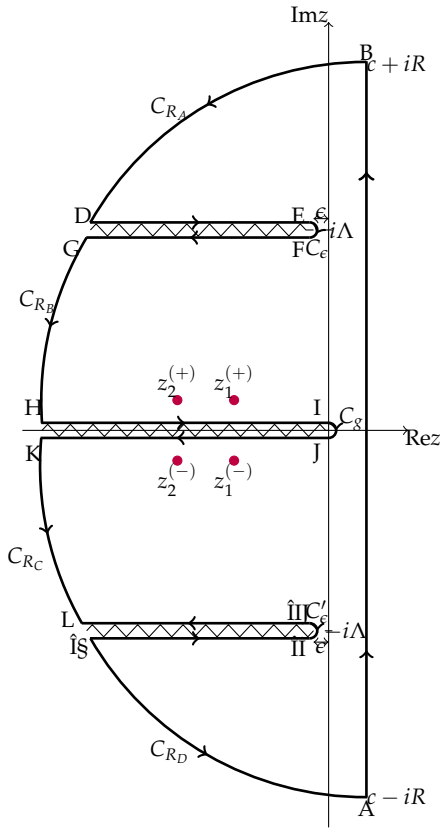


FIGURE A.1: Bromwich contour, brunch cut and poles of the Laplace transformed diagonal and non diagonal elements of $u_{rr'}(t)$. Integration is along a straight line from $c - i\infty$ to $c + i\infty$, where c is a real constant larger than the real part of the poles of the integrand. The contour is closed by a semicircle of radius $R \to \infty$.

where

$$\gamma_0(z^+) - \gamma_0(z^-) = -i\frac{\lambda^2}{8\pi} \tag{A.15}$$

$$\gamma_0(z^+) - \gamma_0(z^-) = -i\frac{\lambda^2}{8\pi}\frac{\sinh(rs)}{rs} \tag{A.16}$$

Therefore

$$I_+(t) = -i\frac{\lambda^2}{4\pi}\int_0^\infty ds e^{-st}\frac{1+\frac{\sinh(rs)}{rs}}{(s^2+\Omega^2+2F(s)+2G(s))^2+\left(\frac{\lambda^2}{8\pi}\right)^2\left(1+\frac{\sinh(rs)}{rs}\right)^2}$$

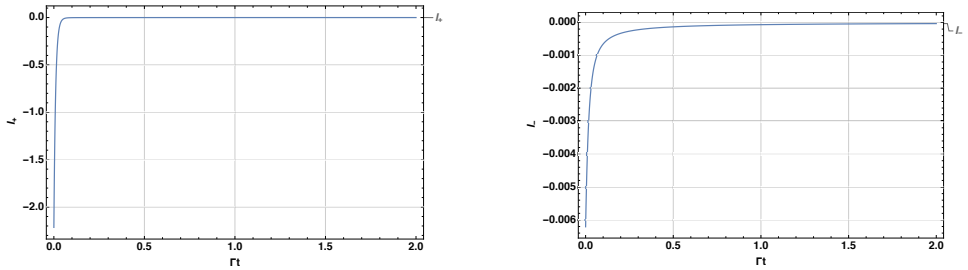
and

$$I_-(t) = -i\frac{\lambda^2}{4\pi}\int_0^\infty ds e^{-st}\frac{1-\frac{\sinh(rs)}{rs}}{(s^2+\Omega^2+2F(s)-2G(s))^2+\left(\frac{\lambda^2}{8\pi}\right)^2\left(1+\frac{\sinh(rs)}{rs}\right)^2}$$

In fig. (A.2) we can see the evolution of the following terms:

1. *Non-Markovian term* $I_{\pm}(t)$,
2. *Markovian term* $f^0_{\pm}(t)$,
3. *and the total term* $u_{11}(t)$, $u_{12}(t)$.

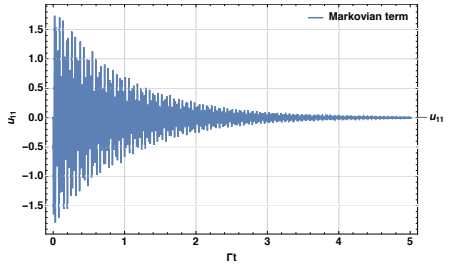
for the parameter $\Omega r = 100$.



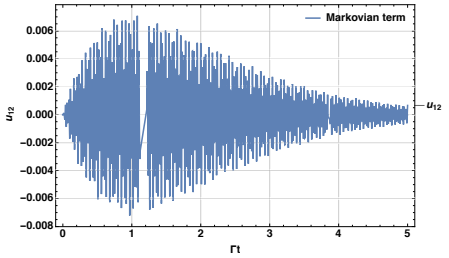
(A) Non-Markovian term $\Omega I_+(t)$ as a function of $\Gamma_0 t$, for $\Omega r = 100$

(B) Non-Markovian term $\Omega I_-(t)$ as a function of $\Gamma_0 t$, for $\Omega r = 100$

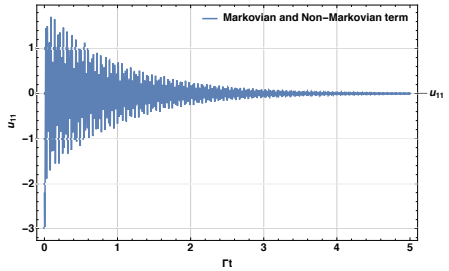
FIGURE A.2: Time evolution of the $\Omega I_{\pm}(t)$, as a function of $\Gamma_0 t$ and for value of $\Omega r = 100$.



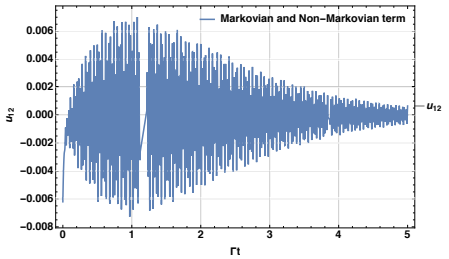
(A) Markovian term $f_+^0(t)$ as a function of $\Gamma_0 t$, for $\Omega r = 100$



(B) Markovian term $f_-^0(t)$ as a function of $\Gamma_0 t$, for $\Omega r = 100$



(C) Markovian and Non-Markovian term u_{11} as a function of $\Gamma_0 t$ for $\Omega r = 100$.



(D) Markovian and Non-Markovian term u_{12} as a function of $\Gamma_0 t$ for $\Omega r = 100$.

FIGURE A.3: Time evolution of the $u_{11}(t)$,$f_+^0(t)$,$f_-^0(t)$, $u_{12}(t)$ as a function of $\Gamma_0 t$ and for value of $\Omega r = 100$.

Appendix B

Exact solution for the system of two harmonic oscillators

We showed that the reduced density matrix propagator for this model is fully determined by the matrices R(t) and S(t). In chapter 4 , we evaluated R(t) and showed its non-Markovian behavior for $\Gamma r \geq 1$. When evaluating the matrix elements $S_{ab}(t)$, we fnd that even for the non-diagonal elements the dominant contribution comes from the functions $u_{11}(t)$ and $u_{22}(t)$ and their derivatives. These functions as we can see at the following plot, are well described by the pole term for very long times.

B.1 Correlation function of harmonic oscillators in a thermal state at temperature T=0, $S_{X_r X_{r'}}$

We begin from the definition of the correlation function $S(t)$, then we present the asymptotic expansion of them and finally we can see the evolution of them at the corresponding plot.

$$S_{X_r X_{r'}} = \left\langle \sum_q \frac{1}{M_q} \int_0^t ds u_{rq}(t-s) \sum_i c_{ir} \hat{q}_i^0(s) \sum_{q'} \frac{1}{M_{q'}} \int_0^t ds' u_{r'q'} \sum_j c_{jr'} \hat{q}_j^0(s') \right\rangle$$

$$S_{X_r X_{r'}} = \sum_q \frac{1}{M_q} \int_0^t ds u_{rq}(t-s) \sum_i c_{ir} \sum_{q'} \frac{1}{M_{q'}} \int_0^t ds' u_{r'q'}(t-s) \sum_j c_{jr'} \left\langle \hat{q}_i^0(s) \hat{q}_j^0(s') \right\rangle$$

The correlation functions for the case of harmonic oscillators in a thermal state with temperature T, can be written as:

$$\left\langle \hat{q}_i^0(s) \hat{q}_j^0(s') \right\rangle_T = \delta_{ij} \frac{1}{2 m_i \omega_i} \coth\left(\frac{\omega_i}{2T}\right) \cos\left[\omega_i (s - s')\right] \tag{B.1}$$

$$\begin{aligned} S_{X_r X_{r'}} &= \sum_{qq'} \frac{1}{M_q M q'} \int_0^t ds \int_0^t ds' u_{rq}(t-s) u_{r'q'}(t-s) \\ &\times \sum_i \frac{c_{ir} c_{ir'}}{2 m_i \omega_i} \coth\left(\frac{\omega_i}{2T}\right) \cos[\omega_i (s-s')] \end{aligned}$$

where the noise kernel, can be defined as

$$v_{rr'}(s) = \sum_i \frac{c_{ir} c_{ir'}}{2 m_i \omega_i} \coth\left(\frac{\omega_i}{2T}\right) \cos(\omega_i s) \tag{B.2}$$

with asyptotic form:

$$S_{X_r X_{r'}} = \sum_{qq'} \frac{1}{M_q M_{q'}} \int_0^t ds \int_0^t ds' u_{rq}(t-s) v_{rr'}(s-s') u_{r'q'}(t-s) \tag{B.3}$$

$$v_{rr'}(s) = \sum_i \frac{c_{ir} c_{ir'}}{2 m_i \omega_i} \coth\left(\frac{\omega_i}{2T}\right) \cos(\omega_i s) \tag{B.4}$$

In our model we consider taht $T = 0 \Rightarrow \coth\left(\frac{\omega_i}{2T}\right) \to 1$ *(vacuum).*
Finally, the correlation matrix take the form:

$$\begin{aligned} S_{X_1 X_1} &= \frac{\lambda^2}{8\pi} \int_0^t ds [u_{11}(s)]^2 + \frac{\lambda^2}{8\pi} \int_0^t ds [u_{12}(s)]^2 \\ &+ \frac{\lambda^2}{16\pi r} \left(\int_0^t ds \int_0^t ds' u_{12}(s) (H[(s-s')+r] - H[(s-s')-r]) u_{11}(s') \right. \\ &+ \left. \int_0^t ds \int_0^t ds' u_{11}(s) (H[(s-s')+r] - H[(s-s')-r]) u_{21}(s') \right) \end{aligned} \tag{B.5}$$

$$\begin{aligned} S_{X_1 X_1} &= \frac{\lambda^2}{16\pi} \Big\{ \frac{1}{\Gamma_+ (\Omega_+^2 + \Gamma_+^2)} + \frac{1}{\Gamma_- (\Omega_-^2 + \Gamma_-^2)} + \frac{\Gamma_+}{2\Omega_+^2 (\Omega_+^2 + \Gamma_+^2)} \\ &- \frac{\Gamma_-}{2\Omega_-^2 (\Omega_-^2 + 4\Gamma_-^2)} - \frac{1}{2r} \frac{\sin(\Omega_+ r)}{\Omega_+^3 \Gamma_+} + \frac{1}{2r} \frac{\sin(\Omega_- r)}{\Omega_-^3 \Gamma_-} \Big\} \end{aligned} \tag{B.6}$$

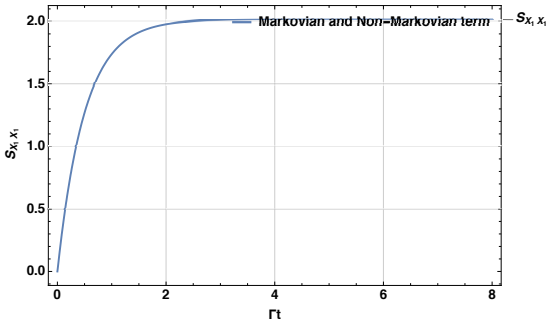
B.2 Correlation function of harmonic oscillators in a thermal state at temperature T=0 , $S_{X_1 X_2}$

We continue with the matrix $S_{X_1 X_2}$*, that calculating with the same way as the matrix* $S_{X_1 X_1}$*.*

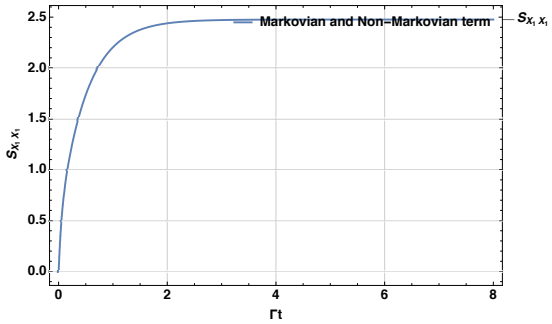
$$S_{X_1 X_2} = S_{X_2 X_1} \tag{B.7}$$

The final expansion is:

$$S_{X_1X_2} = \frac{\lambda^2}{16\pi}\Big\{\frac{1}{\Gamma_+(\Omega_+^2+\Gamma_+^2)}+\frac{1}{\Gamma_-(\Omega_-^2+\Gamma_-^2)}+\frac{\Gamma_+}{2\Omega_+^2(\Omega_+^2+\Gamma_+^2)} + \frac{\Gamma_-}{2\Omega_-^2(\Omega_-^2+4\Gamma_-^2)}-\frac{1}{2r}\frac{\sin(\Omega_+ r)}{\Omega_+^3\Gamma_+}+\frac{1}{2r}\frac{\sin(\Omega_- r)}{\Omega_-^3\Gamma_-}\Big\} \tag{B.8}$$

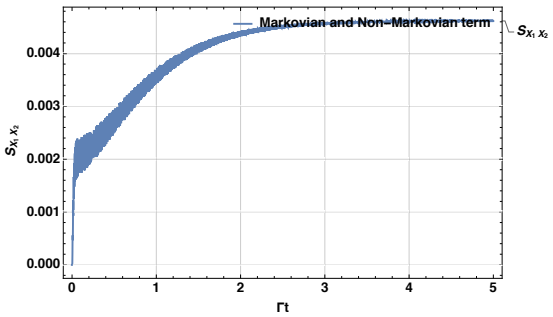


(A) Correlation function $S_{X_1X_1}$,for $r = 100$

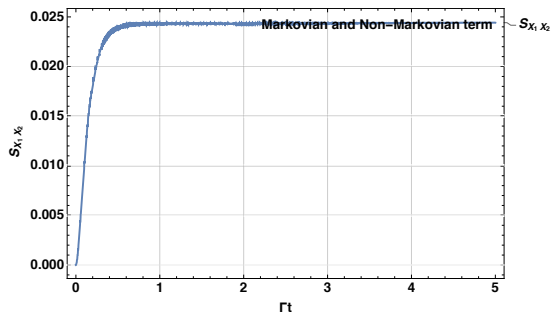


(B) Correlation function $S_{X_1X_1}$, for $r = 1000$

FIGURE B.1: (a)$S_{X_1X_1}$ for parameters of our system $\Gamma r = 0.1$, $a = 10^{-3}$, $\Omega = 1.0$, $\Lambda = 100$ and (b) $S_{X_1X_1}$ for parameters of our system $\Gamma r = 1$, $a = 10^{-3}$,$\Omega = 1.0$, and $\Lambda = 100$



(A) Correlation function $S_{X_1X_2}$, for $r = 100$



(B) Correlation function $S_{X_1X_2}$, for $r = 1000$

FIGURE B.2: (a)$S_{X_1X_2}$ for parameters of our system $\Gamma r = 0.1$, $a = 10^{-3}$, $\Omega = 1.0$, and $\Lambda = 100$ and (b) $S_{X_1X_2}$ for parameters of our system $\Gamma r = 1$, $a = 10^{-3}$, $\Omega = 1.0$, and $\Lambda = 100$

B.3 Correlation function of harmonic oscillators in a thermal state at temperature T=0, $S_{P_1P_1}$

The matrices $S_{P_1P_1}$ and $S_{P_1P_{12}}$ can be defined as

$$S_{P_rP_{r'}} = \langle P_rP_{r'}\rangle = \langle M_r\dot{\hat{X}}_rM_{r'}\dot{\hat{X}}_{r'}\rangle$$

$$S_{P_rP_{r'}} = M_r M_{r'} \langle \dot{\hat{X}}_r \dot{\hat{X}}_{r'} \rangle$$

$$\begin{aligned} S_{P_rP_{r'}} &= M_r M_{r'} \left\langle \sum_q \frac{1}{M_q} \int_0^t ds \dot{u}_{rq}(t-s) \sum_i c_{ir} \hat{q}_i^0(s) \right. \\ &\left. \sum_{q'} \frac{1}{M_{q'}} \int_0^t ds' \dot{u}_{r'q'}(t-s') \sum_j c_{jr'} \hat{q}_j^0(s') \right\rangle \end{aligned}$$

$$\begin{aligned} S_{P_rP_{r'}} &= M_{r'} M_{r'} \sum_q \frac{1}{M_q} \int_0^t ds \dot{u}_{rq}(t-s) \sum_i c_{ir} \\ &\sum_{q'} \frac{1}{M_{q'}} \int_0^t ds' \dot{u}_{r'q'}(t-s') \sum_j c_{jr'} \left\langle \hat{q}_i^0(s) \hat{q}_j^0(s') \right\rangle \end{aligned}$$

$$\left\langle \hat{q}_i^0(s) \hat{q}_j^0(s') \right\rangle = \delta_{ij} \frac{1}{2m_i\omega_i} \coth(\frac{\omega_i}{2T}) \cos\left[\omega_i(s-s')\right]$$

However

$$\begin{aligned} S_{P_rP_{r'}} &= M_r M_{r'} \sum_{qq'} \frac{1}{M_q M_{q'}} \int_0^t ds \int_0^t ds' \dot{u}_{rq}(t-s) \dot{u}_{r'q'}(t-s') \\ &\times \sum_i \frac{c_{ir}c_{jr'}}{2m_i\omega_i} \coth\left(\frac{\omega_i}{2T}\right) \cos\left[\omega_i(s-s')\right] \end{aligned}$$

where $v_{rr'}(s)$ is the noise kernel, defined by:

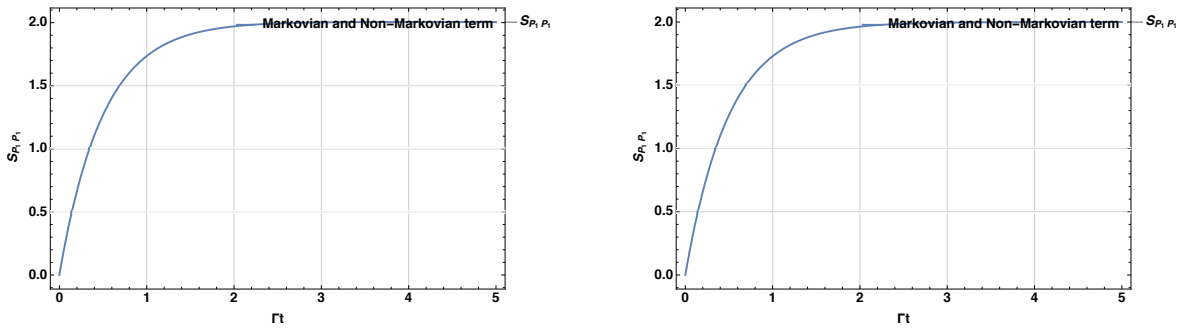
$$v_{rr'}(s) = \sum_i \frac{c_{ir}c_{ir'}}{2m_i\omega_i} \coth\left(\frac{\omega_i}{2T}\right) \cos\left[\omega_i(s-s')\right]$$

therefore:

$$S_{P_rP_{r'}} = M_r M_{r'} \sum_{qq'} \frac{1}{M_q M_{q'}} \int_0^t ds \int_0^t ds' \dot{u}_{rq}(t-s) v_{rr'}(t-s) \dot{u}_{r'q'}(t-s') \quad \text{(B.9)}$$

with asymptotic expansion:

$$\begin{aligned} S_{P_1P_1} &= \frac{\lambda^2}{16\pi} \Big\{ \frac{1}{\Gamma_+} + \frac{1}{\Gamma_-} - \frac{\Gamma_+}{2(\Omega_+^2 + \Gamma_+^2)} + \frac{\Gamma_-}{2(\Omega_-^2 + \Gamma_-^2)} \\ &- \frac{\sin(\Omega_+ r)}{2\Omega_+\Gamma_+ r} + \frac{\sin(\Omega_- r)}{2\Omega_-\Gamma_- r} + \frac{\Gamma_+^3}{2\Omega_+^2(\Omega_+^2 + \Gamma_+^2)} \\ &- \frac{\Gamma_-^3}{2\Omega_-^2(\Omega_-^2 + \Gamma_-^2)} - \frac{\Gamma_+ \sin(\Omega_+ r)}{2\Omega_+^3} + \frac{\Gamma_- \sin(\Omega_- r)}{2\Omega_-^3} \Big\} \end{aligned} \quad \text{(B.10)}$$



(A) Correlation function $S_{P_1P_1}$,for $r = 100$

(B) Correlation function $S_{P_1P_1}$, for $r = 1000$

FIGURE B.3: (a)$S_{P_1P_1}$ for parameters of our system $\Gamma r = 0.1, a = 10^{-3}, \Omega = 1.0, \Lambda = 100$ and (b) $S_{P_1P_1}$ for parameters of our system $\Gamma r = 1, a = 10^{-3}, \Omega = 1.0, \Lambda = 100$

and

$$\begin{aligned} S_{P_1P_2} &= \frac{\lambda^2}{16\pi}\Big\{\frac{1}{\Gamma_+} - \frac{1}{\Gamma_-} + \frac{\Gamma_+}{2(\Omega_+^2+\Gamma_+^2)} + \frac{\Gamma_-}{2(\Omega_-^2+\Gamma_-^2)} \\ &- \frac{\sin(\Omega_+ r)}{2\Omega_+\Gamma_+ r} - \frac{\sin(\Omega_- r)}{2\Omega_-\Gamma_- r} + \frac{\Gamma_+^3}{2\Omega_+^2(\Omega_+^2+\Gamma_+^2)} \\ &+ \frac{\Gamma_-^3}{2\Omega_-^2(\Omega_-^2+\Gamma_-^2)} - \frac{\Gamma_+\sin(\Omega_+ r)}{2\Omega_+^3} - \frac{\Gamma_-\sin(\Omega_- r)}{2\Omega_-^3}\Big\} \end{aligned} \tag{B.11}$$

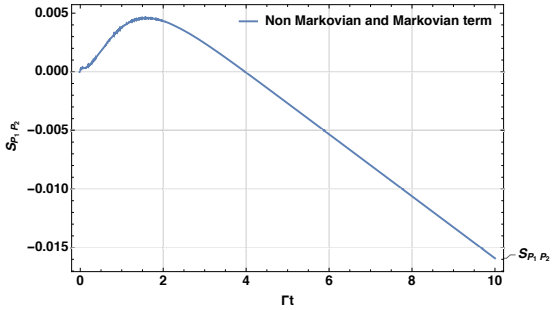
B.4 Correlation function of harmonic oscillators in a thermal state at temperature T=0 , $S_{X_r P_{r'}}$

Finally we define the matrix $S_{X_r P_{r'}}$ as:

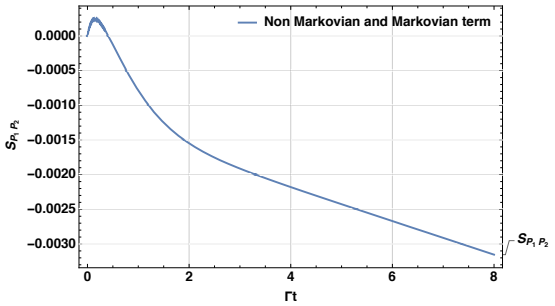
$$S_{X_r P_{r'}} = \langle \hat{X}_r \hat{P}_{r'} \rangle = \langle \hat{X}_r M_{r'} \dot{\hat{X}}_{r'} \rangle = M_{r'} \langle \hat{X}_r \dot{\hat{X}}_{r'} \rangle$$

$$\begin{aligned} S_{X_r P_{r'}} &= M_{r'} \Big\langle \sum_q \frac{1}{M_q} \int_0^t ds u_{rq}(t-s) \sum_i c_{ir} \hat{q}_i^0(s) \\ &\sum_{q'} \frac{1}{M_{q'}} \int_0^t ds' \dot{u}_{r'q'}(t-s') \sum_j c_{jr'} \hat{q}_j 0(s') \Big\rangle \end{aligned}$$

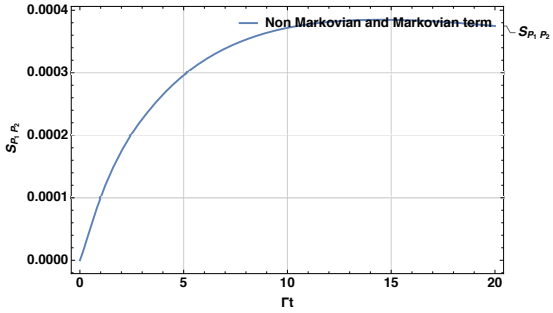
$$\begin{aligned} S_{X_r P_{r'}} &= M_{r'} \sum_{qq'} \frac{1}{M_q M_{q'}} \int_0^t ds u_{rq}(t-s) \sum_i c_{ir} \\ &\sum_{q'} \frac{1}{M_{q'}} \int ds' \dot{u}_{r'q'}(t-s') \sum_j c_{jr'} \langle \hat{q}_i^0(s) \hat{q}_j^0(s') \rangle \end{aligned}$$



(A) Correlation function $S_{P_1P_2}$,for $r = 100$



(B) Correlation function $S_{P_1P_2}$, for $r = 1000$



(C) Correlation function $S_{P_1P_2}$, for $r = 1000$

FIGURE B.4: (a)$S_{P_1P_2}$ for parameters of our system $\Gamma r = 0.1$, $a = 10^{-3}$, $\Omega = 1.0$, $\Lambda = 100$, (b) $S_{P_1P_2}$ for parameters of our system, $\Gamma r = 1$, $a = 10^{-3}$, $\Omega = 1.0$, $\Lambda = 100$

and (c) $S_{P_1P_2}$ for parameters of our system, $\Gamma r = 50$, $a = 10^{-3}$, $\Omega = 1.0$, $\Lambda = 100$

$$\langle \hat{q}_i^0(s)\hat{q}_j^0(s')\rangle = \delta_{ij}\frac{1}{2m_i\omega_i}\coth\left(\frac{\omega_i}{2T}\right)\cos\left[\omega_i(s-s')\right],$$

Therefore:

$$\begin{aligned} S_{X_rP_{r'}} &= M_{r'}\sum_{qq'}\frac{1}{M_qM_{q'}}\int_0^t dsu_{rq}(t-s)\sum_i c_{ir'}\int_0^t ds'\dot{u}_{r'q'}(t-s') \\ &\times \sum_j c_{jr'}\delta_{ij}\frac{1}{2m_i\omega_i}\coth\left(\frac{\omega_i}{2T}\right)\cos[\omega_i(s-s')] \end{aligned}$$

$$S_{X_rP_{r'}} = M_{r'}\sum_{qq'}\frac{1}{M_qM_{q'}}\int_0^t ds\int_0^t ds'u_{rq}(t-s)v_{rr'}(s-s')\dot{u}_{r'q'}(t-s'), \quad \text{(B.12)}$$

where

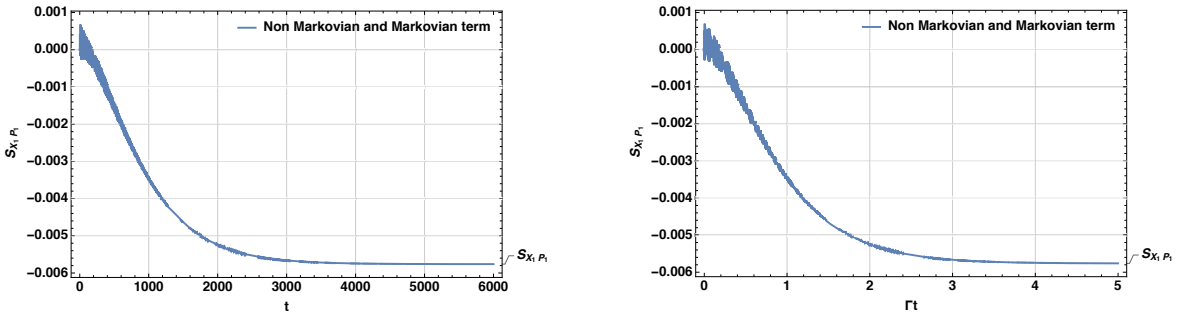
$$v_{rr'}(s) = \sum_i \frac{c_{ir}c_{ir'}}{2m_i\omega_i}\coth\left(\frac{\omega_i}{2T}\right).cos(\omega_i s)$$

Finally:

$$\langle \hat{X}_{r'} \hat{P}_{r'} \rangle = M_{r'} \sum_{qq'} \frac{1}{M_q M_{q'}} \int_0^t ds \int_0^t ds' u_{rq}(t-s) v_{rr'}(s-s') \dot{u}_{r'q'}(t-s')$$

with asymptotic expansion of the correlation functions $S_{X_1 P_1}$, $S_{X_1 P_2}$, the:

$$\begin{aligned} S_{X_1 P_1} &= \frac{\lambda^2}{16\pi} \Big\{ -\frac{1}{2(\Omega_+^2 + \Gamma_+^2)} + \frac{1}{2(\Omega_-^2 + \Gamma_-^2)} - \frac{\Gamma_+^2}{2\Omega_+^2 (\Omega_+^2 + \Gamma_+^2)} \\ &+ \frac{\Gamma_-^2}{2\Omega_-^2 (\Omega_-^2 + \Gamma_-^2)} - \frac{\sin(\Omega_- r)}{2r\Omega_- \Omega_-^2} + \frac{\sin(\Omega_+ r)}{2r\Omega_+ \Omega_+^2} \Big\} \end{aligned} \tag{B.13}$$

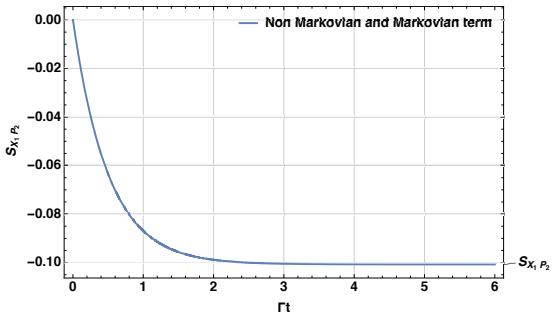


(A) Correlation function $S_{X_1 P_1}$,for $r = 100$

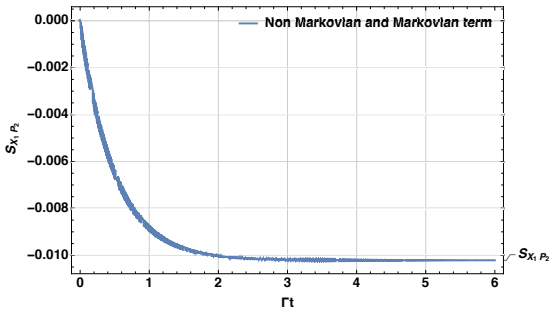
(B) Correlation function $S_{X_1 P_1}$, for $r = 1000$

FIGURE B.5: (a)$S_{X_1 P_1}$ for parameters of our system $\Gamma r = 0.1, a = 10^{-3}, \Omega = 1.0, \Lambda = 100$ and (b) $S_{X_1 P_1}$ for parameters of our system, $\Gamma r = 1, a = 10^{-3}, \Omega = 1.0, \Lambda = 100$

$$\begin{aligned} S_{X_1 P_2} &= \frac{\lambda^2}{16\pi} \Big\{ -\frac{1}{2(\Omega_+^2 + \Gamma_+^2)} - \frac{1}{2(\Omega_-^2 + \Gamma_-^2)} - \frac{\Gamma_+^2}{2\Omega_+^2 (\Omega_+^2 + \Gamma_+^2)} \\ &- -\frac{\Gamma_-^2}{2\Omega_-^2 (\Omega_-^2 + \Gamma_-^2)} + \frac{\sin(\Omega_- r)}{2r\Omega_- \Omega_-^2} + \frac{\sin(\Omega_+ r)}{2r\Omega_+ \Omega_+^2} \Big\} \end{aligned} \tag{B.14}$$



(A) Correlation function $S_{X_1P_2}$,for $r = 100$



(B) Correlation function $S_{X_1P_2}$, for $r = 1000$

FIGURE B.6: (a)$S_{X_1P_2}$ for parameters of our system, $\Gamma r = 0.1, a = 10^{-3}, \Omega = 1.0, \Lambda = 100$ and (b) $S_{X_1P_2}$ for parameters of our system , $\Gamma r = 1, a = 10^{-3}, \Omega = 1.0, \Lambda = 100$

Appendix C

Two-point correlation matrix

Proof of the equation (4.10)

The two-point correlation matrix in phase space coordinates, can be calculated as:

$$V_{ab} = \frac{1}{2}Tr[\hat{\rho}(\hat{\xi}_a\hat{\xi}_b + \hat{\xi}_b\hat{\xi}_a)] - Tr(\hat{\rho}\hat{\xi}_a)Tr(\hat{\rho}\hat{\xi}_b) \tag{C.1}$$

$$V_t = R(t)V_0R^T(t) + S(t) \tag{C.2}$$

Proof.

$$V_{ab} = \frac{1}{2}Tr[\hat{\rho}(\hat{\xi}_a\hat{\xi}_b + \hat{\xi}_b\hat{\xi}_a)] - Tr(\hat{\rho}\hat{\xi}_a)Tr(\hat{\rho}\hat{\xi}_b) \tag{C.3}$$

$$V_{ab} = V_{ab} = \frac{1}{2}Tr[\rho(\xi_a\xi_b + \xi_b\xi_a)] - Tr(\rho\xi_a)Tr(\rho\xi_b) \tag{C.4}$$

We know that the expectation value of an observable O depending on the system variables, can be expressed from the expansion:

$$\langle O(t)\rangle = Tr(O\rho(t)) \tag{C.5}$$

In the eq.(C.4) the trace of the corresponding variables, takes the form:

$$Tr[\rho(\xi_a\xi_b + \xi_b\xi_a)] = Tr[(\xi_a\xi_b + \xi_b\xi_a)\rho] = \langle\xi_a\xi_b + \xi_b\xi_a\rangle \tag{C.6}$$

$$Tr(\rho\xi_a) = Tr(\xi_a\rho) = \langle\xi_a\rangle \tag{C.7}$$

$$Tr(\rho\xi_b) = Tr(\xi_b\rho) = \langle\xi_b\rangle \tag{C.8}$$

Finally, we have:

$$\mathrm{V}_{ab} = \tfrac{1}{2}\langle\xi_a\xi_b + \xi_b\xi_a\rangle - \langle\xi_a\rangle\langle\xi_b\rangle, \text{ with } \quad \langle\xi_a\rangle = 0 \text{ and } \langle\xi_b\rangle = 0 \tag{C.9}$$

Also, we can use the bellow equation in Wigner distribution:

$$\langle\cdots\rangle_{X_i,P_i} \equiv \int_{-\infty}^{+\infty} dX_i \int_{-\infty}^{+\infty} dP_i \cdots W_r(X_i, P_i, t)$$

(C.11)

$$\langle \xi_a \xi_b + \xi_b \xi_a \rangle = \int_{-\infty}^{+\infty} d\xi (\xi_a \xi_b + \xi_b \xi_a) W_t(\xi) \tag{C.12}$$

$$W_t(\xi) = \int \frac{d^{2n}\xi_0}{(2\pi)^N} K_t(\xi_f, \xi_0) W_0(\xi_0) \tag{C.13}$$

$$K_t(\xi_f, \xi_0) = \frac{\sqrt{detS^{-1}(t)}}{\pi^N} \tag{C.14}$$

$$\times \quad exp\left[-\frac{1}{2}[\xi_f^a - \xi_{cl}^a(t)] S_{ab}^{-1}(t) [\xi_f^b - \xi_{cl}^b(t)]\right] \tag{C.15}$$

with

$$\langle \xi_a \xi_b + \xi_b \xi_a \rangle = \int d\xi (\xi_a \xi_b + \xi_b \xi_a) \frac{\sqrt{detS^{-1}(t)}}{\pi^N} \times e^{-\frac{1}{2}[\xi_f^a - \xi_{cl}^a(t)] S_{ab}^{-1}(t) [\xi_f^b - \xi_{cl}^b(t)]} W_0(\xi_0) \tag{C.16}$$

but,

$$\frac{1}{2} Tr[\hat{\rho}(\hat{\xi}_a \hat{\xi}_b + \hat{\xi}_b \hat{\xi}_a)] = \frac{1}{2} \langle \xi_a \xi_b + \xi_b \xi_a \rangle = \langle \xi_a \xi_a \rangle \tag{C.17}$$

$$\langle \xi_a \xi_b \rangle = \int d\xi (\xi_a \xi_b) \frac{\sqrt{detS^{-1}(t)}}{\pi^N} e^{-\frac{1}{2}[\xi_f^a - R_b^a(t) \xi_f^b] S_{ab}^{-1}(t) [\xi_f^b - R_c^b(t) \xi_0^c]} \tag{C.18}$$

in terms of:

$$\xi_a - R_b^a(t) \xi_0^b = \xi_a' \implies \xi_a = \xi_a' + R_b^a(t) \xi_0^b \tag{C.19}$$

$$\xi_b - R_c^b(t) \xi_0^c = \xi_b' \implies \xi_b = \xi_b' + R_c^b(t) \xi_0^c. \tag{C.20}$$

Finally, the eq.(C.18) takes the form:

$$\begin{aligned}
\langle \xi_a \xi_b \rangle &= \int d\xi (\xi'_a + R^a_b(t)\xi^b_0)(\xi'_b + R^b_c(t)\xi^b_0) \frac{\sqrt{detS^{-1}(t)}}{\pi^N} e^{-\frac{1}{2}\xi^{a\prime} S^{-1}_{ab}(t)\xi'_b} W_0(\xi_0) \\
&= \int d\xi (\xi'_a\xi'_b + \xi'_a R^b_c(t)\xi^b_0 + R^a_b(t)\xi^b_0\xi'_b + R^a_b(t)\xi^b_0 R^b_c(t)\xi^b_0) \\
&\quad \frac{\sqrt{detS^{-1}(t)}}{\pi^N} e^{-\frac{1}{2}\xi^{a\prime} S^{-1}(t)\xi^{b\prime}} W_0(\xi_0) \\
&= \int d\xi (\xi'_a\xi'_b) \frac{\sqrt{detS^{-1}(t)}}{\pi^N} e^{-\frac{1}{2}\xi'_a S^{-1}_{ab}(t)\xi'_b} W_0(\xi_0) \\
&+ \int d\xi (\xi'_a R^b_c(t)\xi^b_0) \frac{\sqrt{detS^{-1}(t)}}{\pi^N} e^{-\frac{1}{2}\xi'_a S^{-1}_{ab}(t)\xi'_b} W_0(\xi_0) \\
&+ \int d\xi (R^a_b(t)\xi^b_0\xi'_b) \frac{\sqrt{detS^{-1}(t)}}{\pi^N} e^{-\frac{1}{2}\xi'_a S^{-1}_{ab}(t)\xi'_b} W_0(\xi_0) \\
&+ \int d\xi (R^a_b(t)\xi^b_0 R^b_c(t)\xi^b_0) \frac{\sqrt{detS^{-1}(t)}}{\pi^N} e^{-\frac{1}{2}\xi'_a S^{-1}_{ab}(t)\xi'_b} W_0(\xi_0) \\
&= I_1 + I_2 + I_3 + I_4 \qquad \text{(C.21)}
\end{aligned}$$

with

$$I_1 = \int d\xi' (\xi'_a\xi'_b) \frac{\sqrt{detS^{-1}(t)}}{\pi^N} e^{-\frac{1}{2}\xi'_a S^{-1}_{ab}(t)\xi'_b} W_0(\xi_0) \qquad \text{(C.22)}$$

Using the Gaussian integrals, which are defined by the following identity:
Gaussian integrals

$$\begin{aligned}
\int x^{k_1} \cdots x^{k_{2N}} exp\Big(-\frac{1}{2} \sum_{i,j=1}^{n} A_{ij} x_i x_j \Big) d^n x = \sqrt{\frac{(2\pi)^n}{detA}} \frac{1}{2^N N!} \\
\times \sum_{\sigma \in S_{2N}} (A^{-1})^{k_{\sigma(1)} k_{\sigma(2)}} \cdots (A^{-1})^{k_{\sigma(2N-1)} k_{\sigma(2N)}} \qquad \text{(C.23)}
\end{aligned}$$

The integrals $I_1(t)$, $I_2(t)$, $I_3(t)$, $I_4(t)$ becomes:

$$I_1 = \sqrt{\frac{(2\pi)^{2N}}{detS^{-1}}} \frac{1}{2^N} S_{ab} = S_{ab} \qquad \text{(C.24)}$$

$$I_2 = \frac{\sqrt{detS^{-1}(t)}}{\pi^N} \int d\xi' (\xi'_a R^b_c(t)\xi^b_0) e^{-\frac{1}{2}\xi'_a S^{-1}_{ab}(t)\xi'_b} W_0(\xi_0) = 0 \qquad \text{(C.25)}$$

$$I_3 = \frac{\sqrt{detS^{-1}(t)}}{\pi^N} \int d\xi' (R^a_b(t)\xi^b_0\xi'_b) e^{-\frac{1}{2}\xi'_a S^{-1}_{ab}(t)\xi'_b} W_0(\xi_0) = 0 \qquad \text{(C.26)}$$

$$\qquad \text{(C.27)}$$

and

$$
\begin{aligned}
I_4 &= \frac{\sqrt{detS^{-1}(t)}}{\pi^N}\int d\xi'(R^a_b(t)\xi^b_0 R^b_c(t)\xi^b_0)e^{-\frac{1}{2}\xi'_a S^{-1}_{ab}(t)\xi'_b}W_0(\xi_0) && \text{(C.28)}\\
&= \frac{\sqrt{detS^{-1}(t)}}{\pi^N}\sqrt{\frac{(2\pi)^{2N}}{detS^{-1}}}\frac{1}{2^N}R^a_b(t)R_{bc}V_{0bc} && \text{(C.29)}\\
&= R_{ab}V_{0bc}R^T_{cb} && \text{(C.30)}\\
&= (RV_0R^T)_{ab} && \text{(C.31)}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\langle\xi_a\xi_b\rangle &= I_1 + I_2 + I_3 + I_4 = S_{ab} + 0 + 0 + (RV_0R^T)_{ab} && \text{(C.32)}\\
\langle\xi_a\xi_b\rangle &= (RV_0R^T)_{ab} + S_{ab} && \text{(C.33)}\\
V_{ab} &= (RV_0R^T)_{ab} + S_{ab} && \text{(C.34)}
\end{aligned}
$$

□

Therefore, we find that the two point correlation matrix $V(t)$ at time t, can be written as:

$$V(t) = R(t)V_0R^T(t) + S(t) \qquad \text{(C.35)}$$

where V_0 the correlation function of initial state.

The first term of the right side of the eq. (C) describes the evolution of the system according to the classical equations of motion. On the other hand the second term corresponds to effects of the environment to the system and therefore would be contains all the information that the system loses and trasmitt to the surrounding environment.

Appendix D

Exact solutions to the open system dynamics

D.1 The case of the system of one harmonic oscillator

In this section, all the expressions expressed initially (in Sec. 3.2), will be used in the case of a system consisting of a harmonic oscillator and which interacts with an environment that we consider to be a scaled field. Initially, we should construct our model, that is, define it the Hamiltonian of the total system. So, we consider a system of one harmonic oscillator of mass M_r and of frequency Ω_r that interacts with a environment, that modelled as harmonic oscillator. The total system describes from the following Hamiltonian:

$$H_t = H_{syst} + H_{env} + H_{int} \tag{D.1}$$

where

$$H_{syst} = \frac{1}{2m}p_1^2 + \frac{1}{2}m\omega_1 q_1^2 \tag{D.2}$$

$$H_{env} = \int d^n x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right] = \int d^n k \omega_k \alpha_k^\dagger \alpha_k \tag{D.3}$$

$$H_{int} = \lambda\left(\int d^n x \phi(x) q_1 \delta(x - x_1)\right) = \lambda \sum_{\vec{k}} X_1 \frac{1}{\sqrt{2\omega_k}} q_k e^{ikx_1} \tag{D.4}$$

D.1.1 Calculation of dissipation kernel

The matrix $\gamma_{rr'}$ is the dissipation kernel and is defined by

$$\gamma_{rr'}(s) = -\sum \frac{c_{ir}^* c_{ir'}}{2m_i\omega_i}\sin(\omega_i s) \tag{D.5}$$

Having compared the Eq.(D.4) with the following equation

$$H_{int} = \sum_i \sum_r c_{ir}\hat{X}_r\hat{q}_i,\ r = 1, 2$$

we find,

$$c_{ir} \rightarrow \left\{ \begin{array}{ll} c_{\vec{k}_1} & = \frac{\lambda}{\sqrt{2\omega_k}} e^{\imath k x_1} \end{array} \right.$$

therefore, for the system of one harmonic oscillator and for $m = 1$, the Eq. (D.5) can de written as

$$\begin{aligned} \gamma_{11}(s) &= -\lambda^2 \int \frac{d^n k}{(2\pi)^n} \frac{1}{\sqrt{2\omega_k}} \frac{1}{\sqrt{2\omega_k}} \frac{1}{2\omega_k} \sin(\omega_k s) \\ \gamma_{11}(s) &= -\frac{\lambda^2}{4} \int \frac{d^n k}{(2\pi)^n} \frac{1}{2\omega_k^2} sin(\omega_k s) \end{aligned} \tag{D.6}$$

. The equation (D.6) for 3 dimensions can be written as:

$$\gamma_{11}(s) = -\frac{\lambda^2}{8\pi^2} \int_0^\infty dk \sin(ks) \tag{D.7}$$

and for 1 dimension, the dissipation kernel $\gamma_{11}(s)$ with $\omega_k = |\vec{k}|$, can be written as:

$$\gamma_{11}(s) = -\frac{\lambda^2}{8\pi} \int_0^\infty \frac{\sin(ks)}{k^2} dk \tag{D.8}$$

D.2 Definition of matrix $\tilde{A}_{11}^{-1}(z)$

The definition of the solution of homogeneous part of equation of motion requires the calculation of matrix $A(z)$ and then the inverse of it. For this definition, firstly we define the matrix $A_{11}(z)$, as follow:

$$A_{11}(z) = z^2 + \Omega^2 - \frac{\lambda^2}{8\pi^2} \ln\left(1 + \frac{\Lambda^2}{z^2}\right) \tag{D.9}$$

Secondly, we take the inverse of the matrix D.9, this matrix can be written as:

$$A_{11}^{-1}(z) = \frac{1}{z^2 + \Omega^2 - \frac{\lambda^2}{8\pi^2} \ln\left(1 + \frac{\Lambda^2}{z^2}\right)} \tag{D.10}$$

The last step of this process is to compute the inverse Laplace transform. For this calculation, we have to calculate the poles of the function (D.10), ie to solve the equation:

$$z^2 + \Omega^2 - \frac{\lambda^2}{8\pi^2} \ln\left(1 + \frac{\Lambda^2}{z^2}\right) = 0 \tag{D.11}$$

Solving the above equation, using Mathematica, we found a positive root. This root is naturally not acceptable because when an open quantum system interacts with its environment, the system loses energy. To get rid of this positive root, we enter a

small correction on matrix $A_{11}^{-1}(z)$, the quantity ϵ. Hence, the matrix $\tilde{A}^{-1}(z)$ can be written as:

$$\tilde{A}_{11}^{-1}(z) = \frac{1}{z^2 + \Omega^2 - \frac{\lambda^2}{8\pi^2} ln\left(1 + \frac{\Lambda^2}{(z+\epsilon)^2}\right)} \tag{D.12}$$

so that

$$A_{11}^{-1}(t) = f(t) \tag{D.13}$$

in terms of

$$f(t) = \int_{c-i\infty}^{c+i\infty} dz \frac{e^{zt}}{z^2 + \Omega^2 + 2\tilde{\gamma}_{11}(z)} \tag{D.14}$$

To compute f(t) for the $z = -\Gamma - i\Omega_R$ and the $z = -\Gamma + i\Omega_R$ where,

$$\Gamma = \frac{\lambda^2}{16\pi\Omega} \tag{D.15}$$

and

$$\Omega_R = -\frac{\lambda^2}{16\pi^2\Omega} ln\left(\frac{\Lambda^2}{\Omega^2} - 1\right) + \Omega \tag{D.16}$$

D.3 Calculation of solution $u_{rr'}(t)$

The solution of homogeneous part of (4.13), can be expressed as follow

$$u(t) = L^{-1}[A^{-1}(z)] \tag{D.17}$$

In a weak coupling approximation, for the system of one harmonic oscillator we calculate exactly the solution $u_{rr'}(t)$. This solution can be written as,

$$u_{rr'}(t) = -\frac{\lambda^2}{16\pi\Omega^4}\left(\frac{1}{t}\right) + \frac{1}{2\pi\imath}\sum_{\imath} Res(z_\imath) \tag{D.18}$$

where

$$\sum_i Res(z_i) = e^{-\Gamma t}\left\{\frac{\left[\frac{\imath\lambda^2}{2\pi^2\Omega_R} - 4\Gamma\right]\cos[\Omega_R t] + 4\Omega_R \sin[\Omega_R t]}{4\Gamma^2 - \frac{\lambda^2}{16\pi^4\Omega_R^2} - \frac{\imath\lambda^2\Gamma}{\pi^2\Omega_R} + 4\Omega_R^2}\right\}$$

In weak coupling approximation, the above equation was simplified. Therefore,

$$\sum_i Res(z_i) = \frac{e^{-\Gamma t}}{\Omega_R} \sin(\Omega_R t) \tag{D.19}$$

and the solution $u_{rr'}(t)$ take the form [72]

$$u(t) = \underbrace{-\frac{\lambda^2}{16\pi\Omega^4}\left[\frac{1}{t}\right]}_{\text{Non-Markovian term}} + \underbrace{\frac{1}{2\pi\imath}\frac{e^{-\Gamma t}}{\Omega_R}\sin(\Omega_R t)}_{\text{Markovian term}} \tag{D.20}$$

D.4 Noise kernel

The noise kernel for the system of one harmonic oscillator take the form

$$v_{11}(s) = \frac{\lambda^2}{8\pi^2}\int_0^\infty dk \coth\left(\frac{k}{2T}\right)\cos(ks) \tag{D.21}$$

D.5 Correlation functions

To calculate the correlation functions, we use the relation:

$$\langle X_r X_{r'}\rangle = \sum_{qq'}\frac{1}{M_q M_{q'}}\int_0^t ds\int_0^t ds' u_{rq}(s)v_{qq'}(s-s')u_{q'r'}$$

for the case of a system of one harmonic oscillator we have: $v_{qq'}(s-s') = 2M\gamma T\delta(s-s') = v(s-s')$

It follows that:

$$\begin{aligned}
\langle XX\rangle &= \sum_q \frac{1}{M_q M_{q'}}\int_0^t ds\int_0^t ds' u_{rq}(s)v(s-s')u_{qr'}\\
\langle XX\rangle &= \sum_q \frac{1}{M_q^2}\int_0^t ds\int_0^t ds' u_q(s)v(s-s')u_q(s)\\
\langle XX\rangle &= 2M\gamma T\sum_q \frac{1}{M_q^2}\int_0^t ds\int_0^t ds' u_q(s)\delta(s-s')u_q(s')\\
\langle XX\rangle &= 2\gamma T\sum_q \frac{1}{M_q}\int_0^t ds u_q(s')u_q(s')
\end{aligned}$$

Finally,

$$\langle X^2\rangle = 2\gamma T\sum_q \frac{1}{M_q}\int_0^t ds u_q^2(s') \tag{D.22}$$

Also,

$$S_{P_r P_{r'}} = \langle P_r P_{r'} \rangle = M_r M_{r'} \sum_{qq'} \frac{1}{M_q M_{q'}} \int_0^t ds \int_0^t ds' \dot{u}_{rq}(s) v_{qq'}(s-s') \dot{u}_{q'r'}(s')$$

The noise kernel for $N = 1$ is defined as:

$$v_{qq'}(s-s') = v(s-s') = 2M\gamma T\delta(s-s').$$

The equation D.22 with the expansion of the noise kernel takes the form :

$$\begin{aligned}
\langle PP \rangle &= M_r M_{r'} \sum_{qq'} \frac{1}{M_q M_{q'}} \int_0^t ds \int_0^t ds' \dot{u}_q(s) 2Mq\gamma T\delta(s-s') \dot{u}_q(s') \\
\langle PP \rangle &= M_r M_{r'} \sum_{q} \frac{1}{M_q^2} 2Mq\gamma T \int_0^t ds \int_0^t ds' \dot{u}_q(s) \delta(s-s') \dot{u}_q(s') \\
\langle PP \rangle &= M_r M_{r'} \sum_{q} \frac{2\gamma T}{M_q} \int_0^t ds \dot{u}_q(s') \dot{u}_q(s') \\
\langle PP \rangle &= 2\gamma T M_r M_{r'} \sum_{q} \frac{1}{M_q} \int_0^t ds \dot{u}_q(s') \dot{u}_q(s')
\end{aligned}$$

Therefore:

$$S_{PP} = 2\gamma T M_r M_{r'} \sum_{q} \frac{1}{M_q} \int_0^t ds \dot{u}_q^2(s') \tag{D.23}$$

Finally, we have:

$$S_{X_r P_{r'}} = \langle X_r P_{r'} \rangle = M_{r'} \sum_{qq'} \frac{1}{M_q M_{q'}} \int_0^t ds \int_0^t ds' u_{rq}(s) v_{qq'}(s-s') \dot{u}_{q'r'}(s')$$

For $N = 1$:

$$v_{qq'}(s-s') = v(s-s') = 2Mq\gamma T\delta(s-s')$$

and finally:

$$\begin{aligned}
\langle XP \rangle &= M_{r'} \sum_{q} \frac{1}{M_q^2} \int_0^t ds \int_0^t ds' u_q(s) 2Mq\gamma T\delta(s-s') \dot{u}_q(s') \\
\langle XP \rangle &= M_{r'} \sum_{q} \frac{2M_q\gamma T}{M_q^2} \int_0^t ds \int_0^t ds' u_q(s) \delta(s-s') \dot{u}_q(s') \\
\langle XP \rangle &= M_{r'} \sum_{q} \frac{2\gamma T}{M_q} \int_0^t ds u_q(s') \dot{u}_q(s') \\
\langle XP \rangle &= 2\gamma T M_{r'} \sum_{q} \frac{1}{M_q} \int_0^t ds u_q(s') \dot{u}_q(s')
\end{aligned}$$

i.e.,

$$S_{XP} = 2\gamma T M_{r'} \sum_q \frac{1}{M_q} \int_0^t ds u_q(s') \dot{u}_q(s') \tag{D.24}$$

*The equations (D.22) - (D.24) and the classical equations of motion (**??**) determine exactly the Wigner function propagator. At this point it is worth mentioning that for the case of a harmonic oscillator system we have calculated all the necessary expressions that give us information on the interaction of this open system with its environment.*

D.6 Master equation for the system of a harmonic oscillator

In the case of a harmonic oscillator in a thermal bath, in an ohmic environment and at the Fokker-Planck limit (high temperature limit), the Master equation, as noted by Halliwell, Yu [73] has the form:

$$\frac{\partial \tilde{W}}{\partial t} = -\frac{p}{M}\frac{\partial \tilde{W}}{\partial q} + M\Omega_{ren}^2 q \frac{\tilde{W}}{\partial p} + 2\gamma \frac{\partial \tilde{W}}{\partial p} + 2M\gamma T \frac{\partial^2 \tilde{W}}{\partial p^2} \tag{D.25}$$

where

$$\tilde{\Omega}_{ren} = \Omega^2 - 2\gamma\delta(0)$$

and its inverse:

$$\frac{\partial \rho}{\partial t} = -i[H_R, \rho_r] - i\Gamma[x,[p,\rho_r]] - MD_{pp}[x,[x,\rho_r]] - D_{xp}[x,[p,\rho_r]] \tag{D.26}$$

The eq.(D.26) expresses the time evolution of the density matrix, so it is a Master equation.

Appendix E

Special functions

E.1 Cosine integral

E.1.1 Definition

The cosine integral has the form:

$$\begin{aligned} Ci(x) &= -\int_x^\infty \frac{\cos t dt}{t} && \text{(E.1)} \\ &= \gamma + \ln x + \int_0^x \frac{\cos t - 1}{t} dt && \text{(E.2)} \\ &= \frac{1}{2}\left[Ei(ix) - Ei(-ix)\right] && \text{(E.3)} \\ &= -\frac{1}{2}\left[E_1(ix) + E_1(-ix)\right] && \text{(E.4)} \end{aligned}$$

and $Ci(x)$ has the series expansion:

$$Ci(x) = \gamma + \ln x + \sum_{k=1}^{\infty} \frac{(-x^2)^k}{2k(2k!)} \tag{E.5}$$

E.1.2 Asymptoitc expansions

$$\begin{aligned} Ci(x) &= -\gamma - \ln x + \int_0^x \frac{1 - \cos t}{t} dt \\ &\sim -\gamma - \ln x + \frac{x^2}{2 * 2!} - \frac{x^4}{4 * 4!} + \frac{x^6}{6 * 6!} - \frac{x^8}{8 * 8!} + \cdots, \textit{for } x << 1 \end{aligned} \tag{E.7}$$

and

$$\begin{aligned} Ci(x) &= \frac{\cos x}{x}\left(\frac{1!}{x} - \frac{3!}{x^3} + \frac{5!}{x^5} - \cdots\right) \\ &- \frac{\sin x}{x}\left(1! - \frac{2!}{x^2} + \frac{4!}{x^4} - \cdots\right), \textit{for } x >> 1 \end{aligned}$$

(E.9)

where γ : is the Euler-Mascheroni constant, defined by:

$$\begin{aligned} \gamma &= \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right) \\ &= 0.577215664902 \end{aligned} \tag{E.10}$$

E.2 Sine integral

E.2.1 Defintion

The sine integral has the form:

$$Si(x) = \int_0^x \frac{\sin t}{t} dt \tag{E.11}$$

E.2.2 Asymptotic expansion

$$Si(x) \sim \frac{x}{1*1!} - \frac{x^3}{3*3!} + \frac{x^5}{5*5!} - \frac{x^7}{7*7!} + \cdots, for\ x << 1 \tag{E.12}$$

and

$$\begin{aligned} Si(x) &= \frac{\pi}{2} - \frac{\sin x}{x}\left(\frac{1!}{x} - \frac{3!}{x^3} + \frac{5!}{x^5} - \cdots\right) \\ &- \frac{\cos x}{x}\left(1! - \frac{2!}{x^2} + \frac{4!}{x^4} - \cdots\right), for\ x >> 1 \end{aligned} \tag{E.14}$$

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