Syzygies of Ideals of Polynomial Rings over Principal Ideal Domains

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ABSTRACT

We study computational aspects of syzygies of graded modules over polynomial rings $R[w_1, \ldots, w_g]$ when the base R is a discrete valuation ring. In particular, we use the torsion of their syzygies over R to provide a formula which describes the behavior of the Betti numbers when changing the base to the residue field or the fraction field of R. Our work is motivated by the deformation theory of curves.

CCS CONCEPTS

• **Theory of computation → Computational geometry**; • **Mathematics of computing**;

KEYWORDS

Commutive algebra, syzygies, principal ideal domains, reduction, lifting MSC:13D02, 13P20

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1 INTRODUCTION

The study of syzygies of modules is one of the main topics of interest of combinatorial commutative algebra with numerous algorithmic applications. In the context of computational algebraic geometry, usually one studies syzygies of ideals of the polynomial ring $k[w_1, \ldots, w_g]$, where k is a field. However, deformation theory of curves deals with flat families of curves over discrete valuation rings R. In particular, non-hyperelliptic curves of genus g are better understood in terms of their canonical ideal, the ideal in $S = R[w_1, \ldots, w_g]$ that defines the canonical embedding of the family in $\mathbb{P}_R^{g-1}$.

The last two authors have studied the R-modules of relative polydifferentials for certain cyclic covers of the projective line [7], which lead to a description of the relative canonical ideal by the first, second and fourth author in [3]. Applications of syzygies and free resolutions to the study of curves with automorphisms are given by Terezakis, Tsouknidas and the fourth author in [8]. The difference in the behaviour of the Betti numbers in the special and generic fibre is expected to provide new obstructions to the theory of lifting of curves with automorphisms, see [9], [10], since liftings of indecomposable representations of the automorphism group should respect the free and torsion part. Moreover the relative point of view contributes to the understanding of the situation concerning Green's conjecture in positive characteristic, see [2] for a refined version.

Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}_R = \langle x \rangle$, fraction field K and residue field k. Deformation theory and modular representation theory are related to the effect of taking the base to be any of the three rings R, K, k. Let $S = R[w_1, \ldots, w_g]$ and consider an S-module M such that the generator x of $\mathfrak{m}_R$ is not a zero divisor on M. This leads to the study a) of $\widehat{S} = K[w_1, \ldots, w_g]$ and the respective $\widehat{S}$-module $\widehat{M} = M \otimes \widehat{S}$ (corresponding to the *generic fibre*) and b) of $\overline{S} = k[w_1, \ldots, w_g]$ and the respective $\overline{S}$-module $\overline{M} = M \otimes \overline{S}$ (corresponding to the *special fibre*).

Grothendieck's relative point of view leads to the question of how the syzygies and the Betti numbers of the special and the generic fibre of a family are related when considered over k or K or even over R. The study of syzygies becomes automatically more challenging over R since the non-zero elements of the PID may not be invertible and modules might have torsion, see [1, chap. 4] for a more comprehensive account and also [12]. On the other hand simplicial homology over $\mathbb{Z}$ has been extensively studied and techniques have been developed to account for that case and the different behavior over $\mathbb{Q}$, [4]. It is well known that, even in the case of monomial ideals, the minimal free resolution depends on the characteristic of the ground field, the classical example being the triangulation of the projective plane.

Example 1. The Betti numbers of

$B = \langle abc, abf, ace, ahe, ahf, bch, bhe, bef, chf, cef \rangle \lhd k[a, b, c, e, f, h]$

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differ when $\mathrm{char}(k) = 0$ (table on the left) and $\mathrm{char}(k) = 2$ (table on the right).

	0	1	2	3
0	1	0	0	0
1	0	0	0	0
2	0	10	15	6
3	0	0	0	0

	0	1	2	3	4
0	1	0	0	0	0
1	0	0	0	0	0
2	0	10	15	6	1
3	0	0	0	1	0

Thus when $\mathrm{char}(k) = 2$, the ideal B has a third and a fourth graded syzygy of degree 6 which do not appear over characteristic zero (or any other characteristic $p \neq 2$ for that matter), see also [11, Ex. 12.4].

In the sequel we give explicit reasons for this behavior. We consider syzygies of finitely generated graded S-modules of the polynomial ring $S = R[w_1, \ldots, w_g]$. We will see that it makes sense to consider minimal free resolutions of the graded S-module M, and we will define the graded *Betti* numbers of M.

The structure of this paper is as follows: we first discuss minimal free resolutions of graded modules over $R[w_1, \ldots, w_g]$ and Nakayama's lemma (Lemma 2). Using the classification theorem for modules over PIDs, we see how the existence of R-torsion on the syzygies affects the resolution (Theorem 6).

We also explain how torsion can be read from the Smith normal form of the reduced matrix of the differentials (Corollary 7). In the last section we give a detailed computation of Example 1 and conclude with a method (Algorithm 1) which, given the generators of a graded ideal I of $\mathbb{Z}[w_1, \ldots, w_g]$, outputs all primes p for which the Betti numbers of I in $\mathbb{F}_p[w_1, \ldots, w_g]$ differ from the Betti numbers of I in $\mathbb{Q}[w_1, \ldots, w_g]$ and give information for possible obstructions.

2 SYZYGIES OVER GENERAL RINGS

Let $(R, \mathfrak{m}_R)$ be as in the introduction, and let $S = R[w_1, \ldots, w_g]$ be the polynomial ring in g variables, graded by assigning the degree 1 to each w_i, $i = 1, \ldots, g$. Thus $S = \sum_{i \geq 0} S_i$, with $S_0 = R$. We let $\mathfrak{m}$ and $\mathfrak{m}_S$ be respectively the prime and maximal ideals $\mathfrak{m} = \langle w_1, \ldots, w_g \rangle$, $\mathfrak{m}_S = \mathfrak{m} + \mathfrak{m}_R S$ of S. Observe that $k = S/\mathfrak{m}_S = R/\mathfrak{m}_R$.

Let M be a finitely generated graded S-module. Thus we have $M = \sum_{i \geq a} M_i$, where $S_j M_i \subset M_{i+j}$ and in particular M_i is an R-module, for $i \geq a$. Let $m_1, \ldots, m_n$ form a generating set of M. It is clear that $\{\overline{m}_i = m_i + \mathfrak{m}_S M,\ i = 1, \ldots, n\}$ is a generating set of $M/\mathfrak{m}_S M$. The converse, i.e. Nakayama's lemma, holds when the elements m_i are homogeneous. The proof follows the same lines as the standard proof for the graded case, [5, lemma 1.4]. We include it here for completeness of the exposition.

Lemma 2 (Nakayama). *Let M be a finitely generated positively graded S-module, $m_1, \ldots, m_n \in M$ homogeneous so that $\overline{m}_i$, $i = 1, \ldots, n$ generate $M/\mathfrak{m}_S M$. Then $m_1, \ldots, m_n$ generate M.*

Proof. Let $M' = \sum_{i=1}^{n} S m_i$ and consider the finitely generated graded S-module $N = M/M'$. By our assumption on the m_i, $M' + \mathfrak{m}_S M = M$, thus $N/\mathfrak{m}_S N = 0$ and $\mathfrak{m}_S N = N$. If $N \neq 0$, there is a nonzero graded element of least degree in N. Since $\mathfrak{m}_S N = N$, this element must have degree zero. It follows that $N_0 = \mathfrak{m}_R N_0$. Since R is a local PID, Nakayama's lemma in the local case gives that $N_0 = 0$. It follows that $N = 0$ as desired. □

It follows that the least number of homogeneous elements needed to generate M is the dimension of the $S/\mathfrak{m}_S$-vector space $M/\mathfrak{m}_S M$. We proceed to construct a **minimal graded** free resolution of M. Let $m_1, \ldots, m_n$ be a minimal set of homogeneous generators of M. We let F_0 be the free module $F_0 = \bigoplus_i S e_i$ on generators e_i, $\deg(e_i) = \deg(m_i)$ $(i = 1, \ldots n)$ and let $\pi_0 : F_0 \longrightarrow M$ be the epimorphism determined by $\pi_0(e_i) = m_i$. This gives the short exact sequence

$$0 \to \ker \pi_0 \xrightarrow{\iota_0} \mathrm{F}_0 \xrightarrow{\pi_0} \mathrm{M} \to 0,$$

where $\ker\pi_0 \subset \mathfrak{m}_S F_0$. Since $\ker\pi_0$ is a finitely generated graded S-module we repeat this procedure to obtain $\pi_1 : F_1 \longrightarrow \ker \pi_0$ and $\delta_1 : F_1 \longrightarrow F_0$ as the composition $F_1 \xrightarrow{\pi_1} \ker \pi_0 \xrightarrow{\iota_0} F_0$. Note that $\delta_1(F_1) = \ker \pi_0$ and that we have specified the basis of F_1 that maps to a minimal homogeneous generating set of $\ker \pi_0$. Iterating this procedure, we obtain a free graded resolution of M which is minimal since by construction $\ker\pi_i \subset \mathfrak{m}_S F_i$ for all $i \geq 0$:

$$(F_\bullet, \delta_\bullet) : \cdots \longrightarrow F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\pi_0} M \longrightarrow 0$$

For each $i \geq 1$, the resolution above breaks into a short exact sequence

$$0 \to \ker \pi_\mathrm{i} \xrightarrow{\iota_\mathrm{i}} \mathrm{F}_\mathrm{i} \xrightarrow{\pi_\mathrm{i}} \ker \pi_{\mathrm{i}-1} \to 0, \tag{1}$$

$\delta_i : F_i \to F_{i-1}$ being the composition $\iota_{i-1}\pi_i$ for $i \geq 1$. We note that the differentials δ_i are of degree zero, $\delta_i(F_i) \subset \mathfrak{m}_S F_{i-1}$ and that δ_i maps a basis of F_i to a minimal set of homogeneous generators of $\delta_i(F_i)$, as in [5, Corollary 1.5]. We write each F_i as a direct sum, indexed by $\mathbb{Z}$, of copies of S shifted by the degrees of the generators:

$$F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}},$$

where finitely many of the $\beta_{i,j}$ are nonzero. The exponent $\beta_{i,j} \in \mathbb{N}$ that counts the number of minimal generators of degree j in F_i is called the (i, j)-**graded Betti number** of M and equals the dimension $\dim_k \mathrm{Tor}_i^S(k, M)_j$ as in [5, Corollary 1.7]. We write $\beta_{i,j}(M)$ when needed to emphasize the module M. The S-modules

$$\Pi_i = \ker \pi_{i-1} = \ker \delta_{i-1}, \tag{2}$$

for $i \geq 1$, are known as the i-th **syzygies** of M, and we set $\Pi_0 = M$, so that Π_i is a graded S-module for all i. By successively taking homology of the short exact sequences in (1) we get that

$$\mathrm{Tor}_1^S(\Pi_{i-1}, R) = \mathrm{Tor}_i^S(M, R),\ i \geq 1. \tag{3}$$

We let $F_{i,j}$ be the direct summand of F_i at degree j and denote by $\delta_{i,j}$ the restriction

$$\delta_{i,j} = \delta_i|_{F_{i,j}} : F_{i,j} \longrightarrow F_{i-1,j}.$$

Remark 3. Let $F_\bullet$ be a minimal graded free resolution of the graded S-module M. By tensoring $F_\bullet$ with $R = S/\mathfrak{m}$ (over S) we obtain a graded complex of R-modules:

$$F_\bullet \otimes R : \cdots \longrightarrow F_1 \otimes R \xrightarrow{\delta_1 \otimes 1_R} F_0 \otimes R \longrightarrow M \otimes R \longrightarrow 0$$

whose homology at the i-th position is $\mathrm{Tor}_i^S(R, M)$, a graded S-module. Note that, for $\alpha > 0$, $S(-\alpha) \otimes R$ only lives in degree α. Thus, to compute $\mathrm{Tor}_i^S(R, M)_j = \mathrm{Tor}_1^S(R, \Pi_{i-1})_j$ one needs to consider the following s.e.s. derived from (1):

$$0 \to \mathrm{Tor}_1^S(\Pi_i, R)_j \to (\Pi_{i+1})_j \otimes R \to (S(-j)^{\beta_{i,j}} \otimes R)_j \to (\Pi_i)_j \otimes R \to 0. \tag{4}$$

3 SYZYGIES OVER GENERIC AND SPECIAL FIBRES

Let us start by recalling the notation set-up in the introduction: R is a discrete valuation ring with maximal ideal $\mathfrak{m}_R = \langle x\rangle$, K is the fraction field of R and $k = R/\mathfrak{m}_R$ is the residue field. We set $S = R[w_1, \dots, w_g]$, $\mathfrak{m} = \langle w_1, \dots, w_g\rangle \lhd S$ and $\mathfrak{m}_S = \langle x, w_1, \dots, w_g\rangle$. Let M be a finitely generated graded S-module. Note that $\widehat{S} = K[w_1, \dots, w_g]$ is the localization of S at the multiplicatively closed subset R^* and similarly for $\widehat{M} = M \otimes \widehat{S}$. Finally, $\overline{\mathfrak{m}} = \langle w_1, \dots, w_g\rangle \lhd \overline{S}$ is the maximal graded ideal of $\overline{S} = k[w_1, \dots, w_g]$ and $\overline{M} = M/xM$.

We note that if N is a finitely generated R-module, then since R is a local PID, N is a direct sum of the form

$$N = \bigoplus^{\mathrm{rk}(N)} R \oplus \mathrm{tor}(N),$$

where $\mathrm{rk}(N)$ is the rank of N as an R-module, while $\mathrm{tor}(N)$, the torsion part of N, is a direct sum of the form

$$\mathrm{tor}(N) = \bigoplus_{\nu=1}^{t(N)} R/Rx^{a(\nu,N)}, \text{ where } a(\nu, N) \in \mathbb{N}, \text{ for } \nu = 1, \dots, t(N)\,.$$

Observe that $\mathrm{tor}(N)$ is still visible when tensoring with k (special fibre), since $N \otimes_R k = k^{\mathrm{rk}(N)+t(N)}$, while it disappears when tensoring with K (generic fibre), since $N \otimes_R K = K^{\mathrm{rk}(N)}$.

Let M be a finitely generated graded S-module such that x is a not a zero divisor on M, i.e. multipication by x is injective and M is a flat R-module. Let $F_\bullet$ be a minimal free resolution of M. To study M we will tensor $F_\bullet$ with R and get a complex of R-modules.

It is known that under our assumptions, reduction to the special fibre preserves exactness, see for example [11, Thm 20.3]. The short proof is included here for completeness of the exposition.

Lemma 4. *If $F_\bullet$ is a free resolution of M as an S-module, then $F_\bullet \otimes S/xS$ is a free resolution of M/xM as an S/xS-module.*

Proof. The short exact sequences $0 \to S \to S \to S/xS \to 0$ and $0 \to M \to M \to M/xM \to 0$, imply that $\mathrm{Tor}_i^S(M, S/xS) = 0$, for $i \geq 1$ and thus $F_\bullet \otimes S/xS$ is exact. □

We also note that flatness of K over R implies flatness of the ring $K[w_1, \dots, w_g]$ over S. Thus we have the following:

Lemma 5. *If $F_\bullet$ is a free resolution of M, seen as an S-module then $\widehat{F_\bullet} = F_\bullet \otimes K[w_1, \dots, w_g]$ is a free resolution of $\widehat{M} = M \otimes K[w_1, \dots, w_g]$ as an $K[w_1, \dots, w_g]$-module.*

Let $(F_\bullet, \delta_\bullet)$ be a minimal graded free resolution of the graded S-module M. By Lemma 5, $\widehat{F_\bullet}$ is a graded free resolution of $\widehat{M}$, however it might not be minimal. We write $\Pi_{i,j}$ for the j-th graded piece of $\Pi_i = \ker(\delta_{i-1})$ and $\overline{\Pi}_{i,j}$ for the R-module $\Pi_{i,j} \otimes R$ which we decompose into its cyclic R-components. We will see that the quantities $f_{i,j} := \mathrm{rk}(\overline{\Pi}_{i,j}), t_{i,j} := t(\overline{\Pi}_{i,j})$ and $s_{i,j} = \mathrm{rk}(\mathrm{Tor}_1^S(R, \Pi_i)_j)$ are critical when we measure the difference between the graded Betti numbers of the generic and the special fibre.

Theorem 6. *Let S be $R[w_1, \dots, w_g]$, M be a finitely generated graded S-module which is flat as an R-module, Π_i be the i-th syzygy of M and $t_{i,j}$ be the number of nonzero cyclic summands of $\overline{\Pi}_{i,j}$, for $i \geq 0$.*

(1) $\beta_{i,j}(M) = \beta_{i,j}(\overline{M})$, for $i \geq 0$.
(2) $\beta_{i,j}(M) = \beta_{i,j}(\widehat{M}) + t_{i,j} + t_{i-1,j}$ for $i \geq 1$.

Proof. Let $F_\bullet$ be a minimal graded free resolution of the graded S-module M. By Lemma 4, it follows that $\overline{F}_\bullet = F_\bullet \otimes S/xS$ is a free resolution of $\overline{M}$. Moreover, since $\delta_i(F_i) \subset \mathfrak{m}_S F_{i-1}$, it follows that $\overline{\delta_i}(\overline{F_i}) \subset \overline{\mathfrak{m}}\overline{F}_{i-1}$ and $\overline{F}_\bullet$ is a minimal free resolution of $\overline{M}$. Thus $\beta_{i,j}(M) = \beta_{i,j}(\overline{M})$.

For the generic fibre, by Lemma 5, $\widehat{F} = F_\bullet \otimes K[w_1, \dots, w_g]$ is a free resolution of $\widehat{M}$ and we need to compute $\dim_K \mathrm{Tor}_i^{\widehat{S}}(\widehat{M}, K)_j$. By the Künneth formula [13, Th. 3.6.1], $\mathrm{Tor}_i^{\widehat{S}}(\widehat{M}, K)$ is the localization of $\mathrm{Tor}_i^S(M, R)$ at R^* and

$$\mathrm{Tor}_i^{\widehat{S}}(\widehat{M}, K)_j \cong \mathrm{Tor}_i^S(M, R)_j \otimes K.$$

Thus by (3) it suffices to examine the R-structure of $\mathrm{Tor}_1^S(\Pi_{i-1}, R)_j$. We consider the tensor product $F_\bullet \otimes R$. By (4) we have

$$0 \to \mathrm{Tor}_1^S(\Pi_{i-1}, R)_j \to \overline{\Pi}_{i,j} \longrightarrow R^{\beta_{i-1,j}} \longrightarrow \overline{\Pi}_{i-1,j} \to 0.$$

Since $\overline{\Pi}_{i,j}/\mathrm{Tor}_1^S(\Pi_{i-1}, R)_j \hookrightarrow R^{\beta_{i-1,j}}$, it follows that the quotient $\overline{\Pi}_{i,j}/\mathrm{Tor}_1^S(\Pi_{i-1}, R)_j$ is free and

$$\mathrm{tor}(\mathrm{Tor}_1^S(\Pi_{i-1}, R)_j) = \mathrm{tor}(\overline{\Pi}_{i,j}).$$

Thus

$$\overline{\Pi}_{i,j}/\mathrm{Tor}_1^S(\Pi_{i-1}, R)_j = R^{f_{i,j}-s_{i-1,j}}.$$

By the short exact sequence

$$\begin{array}{ccccccc} 0 \to & \overline{\Pi}_{i,j}/\mathrm{Tor}_1^S(\Pi_{i-1}, R)_j & \to & R^{\beta_{i-1,j}} & \longrightarrow & \overline{\Pi}_{i-1,j} & \longrightarrow 0 \\ & \| & & & & \| & \\ & R^{f_{i,j}-s_{i-1,j}} & & & & R^{f_{i-1,j}} \oplus \mathrm{tor}(\overline{\Pi}_{i-1,j}) & \end{array}$$

we have that

- $\beta_{i-1,j} = f_{i-1,j} + t_{i-1,j}$ (from the epimorphism),
- $\beta_{i-1,j} = (f_{i,j} - s_{i-1,j}) + f_{i-1,j}$ (from the additivity of ranks).

It follows that the rank $s_{i-1,j}$ of $\mathrm{Tor}_1^S(\Pi_{i-1}, R)_j$ is equal to

$$s_{i-1,j} = (f_{i,j} + f_{i-1,j}) - \beta_{i-1,j} = f_{i,j} + (f_{i-1,j} - \beta_{i-1,j}) = (\beta_{i,j} - t_{i,j}) - t_{i-1,j}.$$

We tensor $\mathrm{Tor}_i^S(R, M)_j$ with K to obtain that

$$\beta_{i,j}(\widehat{M}) = s_{i-1,j} = \beta_{i,j}(M) - t_{i,j} - t_{i-1,j}.$$

□

How does one compute $t_{i,j}$? This can be done by computing the Smith normal form of the matrix of differentials $\overline{\delta}_{i,j} = \delta_{i,j} \otimes R$. We proceed as in [4]. Note that $R^{\beta_{i,j}} = F_{i,j} \otimes R$, while $R^{\beta_{i-1,j}} = F_{i-1,j} \otimes R$ and $\overline{\delta}_{i,j} : R^{\beta_{i,j}} \longrightarrow R^{\beta_{i-1,j}}$. Let $B_{i,j}$ be the matrix of $\overline{\delta}_{i,j}$ with respect to the canonical bases of $R^{\beta_{i,j}}$ and $R^{\beta_{i-1,j}}$. There is a change of basis for $R^{\beta_{i,j}}$ and $R^{\beta_{i-1,j}}$ so that the matrix of $\overline{\delta}_{i,j}$ with respect to these new bases is the *Smith normal form* of $B_{i,j}$, say $A_{i,j}$. The Smith normal form $A_{i,j}$ contains an upper left diagonal block

$$\mathrm{diag}\left(b_1, b_2, \dots, b_{t(i-1,j)}\right),$$

with $b_1 | b_2 | \cdots | b_{t(i-1,j)} \neq 0$, while the rest of the blocks of $A_{i,j}$ are zero. We note that since $F_\bullet$ is a minimal resolution, all $b_a \in \mathfrak{m}_R$, for $a = 1, \dots, t(i-1,j)$ and thus $b_a = x^{e(a)}$, for some positive integer

$e(a)$. It is clear that $t(i-1,j)$ is the rank of Im $\overline{\delta}_{i,j}$ and thus the rank of ker $\overline{\delta}_{i,j}$ equals $\beta_{i,j} - t(i-1,j)$. Let us now consider the Smith normal form for $\overline{\delta}_{i+1,j}$. Suppose that its nonzero block is

$$\operatorname{diag}\left(c_1, c_2, \ldots, c_{t(i,j)}\right)$$

and let $\epsilon_1, \ldots, \epsilon_{\beta_{i,j}}$ be the basis of $R^{\beta_{i,j}}$ relative to this normal form. Thus $c_a\epsilon_a \in$ Im $\overline{\delta}_{i+1,j}$. Since $\overline{\delta}_{i,j}\overline{\delta}_{i+1,j} = 0$, we have that $\overline{\delta}_{i,j}(c_a\epsilon_a) = c_a\overline{\delta}_{i,j}(\epsilon_a) = 0$, and we conclude that $\epsilon_1, \ldots, \epsilon_{t(i,j)}$ are in ker $\overline{\delta}_{i,j}$, for $a = 1, \ldots, t(i,j)$. Thus,

$$\operatorname{Tor}_i^S(M,R)_j \cong R^{\beta_{i,j}-t(i-1,j)-t(i,j)} \oplus R/c_1R \oplus \cdots \oplus R/c_{t(i,j)}R.$$

By the uniqueness of the decomposition of $\operatorname{Tor}_i^S(M,R)_j$ and induction on i, it follows that $t(i,j) = t_{i,j}$ for all i. We have shown the following:

Corollary 7. *If $(F_\bullet, \delta_\bullet)$ is a minimal graded free resolution of M over S and the Smith normal form of the matrix of $\overline{\delta}_{a,j}$ has rank $t(a-1,j)$, $a \geq 1$, then $t_{i,j} = t(i,j)$ for $i \geq 0$ and $\beta_{i,j}(\widehat{M}) = \beta_{i,j}(M) - t_{i,j} - t_{i-1,j}$.*

4 EXAMPLE

Let us now return to Example 1. Let $\mathbb{Z}_2$ be the ring of 2-adic integers with fraction field $\mathbb{Q}_2$ and residue field $\mathbb{F}_2$. Let $S = \mathbb{Z}_2[a, \ldots, h]$, $\mathfrak{m} = \langle a, \ldots, h\rangle$, $B = \langle abc, abf, ace, ahe, ahf, bch, bhe, bef, chf, cef\rangle$ and $M = S/B$. We will show that

$$\beta_{0,0}(M) = 1, \beta_{1,3}(M) = 10,$$

$$\beta_{2,4}(M) = 15, \beta_{3,5}(M) = 6, \beta_{3,6}(M) = 1, \beta_{4,6}(M) = 1,$$

and that M has a minimal graded free resolution over S of the form

$$\begin{tikzcd}
 & F_4 \arrow[d, equal] & F_3 \arrow[d, equal] & F_2 \arrow[d, equal] & F_1 \arrow[d, equal] \\
0 \arrow[r] & S(-6) \arrow[r, "\delta_4"] & S(-5)^6 \oplus S(-6) \arrow[r, "\delta_3"] & S(-4)^{15} \arrow[r, "\delta_2"] & S(-3)^{10} \arrow[dllll, rounded corners, "\delta_1"] \\
S = F_0 \arrow[r] & M \arrow[r] & 0 & &
\end{tikzcd} \tag{5}$$

We will show that Π_4, the kernel of $\delta_3 : F_3 \to F_2$, has a minimal generating set of two elements, with the generator of degree 6 becoming torsion in $F_\bullet \otimes_S R \cong F_\bullet \otimes S/\mathfrak{m}$, impling that $t_{4,6} = 1$. This means that for $\widehat{S} = \mathbb{Q}_2[a, \ldots, h]$, we get the following exact diagram:

$$\begin{tikzcd}
0 \arrow[r] & \widehat{S}(-6) \arrow[r, "\cong"] \arrow[d, hook] & \widehat{S}(-6) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \widehat{S}(-6) \arrow[r, "\delta_4"] & \widehat{S}(-5)^6 \oplus \widehat{S}(-6) \arrow[r, "\delta_3"] & \widehat{S}(-4)^{15} \arrow[dlll, rounded corners, "\delta_2"] \\
\widehat{S}(-3)^{10} \arrow[r, "\delta_1"] & \widehat{S} \arrow[r] & \widehat{M} \arrow[r] & 0
\end{tikzcd}$$

The degree 6 elements in both F_4, F_3 have to be removed in order to obtain a minimal free resolution in the generic fibre.

We used Macaulay2 [6] in order to compute the above resolution. The following code

```
T = ZZ[a,b,c,e,f,h]
J = ideal(a*b*c,a*b*f,a*c*e,a*h*e,a*h*f,
     b*c*h,b*h*e,b*e*f,c*h*f,c*e*f)
rs = res J
rs.dd
```

produces the free resolution $G_\bullet$ of T/J over T

$$0 \longrightarrow T^2 \xrightarrow{\theta_4} T^{10} \xrightarrow{\theta_3} T^{17} \xrightarrow{\theta_2} T^{10} \xrightarrow{\theta_1} T, \tag{6}$$

where the differentials θ_3, θ_4 correspond to the matrices (also denoted for simplicity by θ_3, θ_4)

$$\theta_4 = \begin{pmatrix} 0 & f \\ e & 0 \\ -b & 0 \\ -h & 0 \\ 0 & -c \\ -c & 0 \\ 0 & a \\ a & 0 \\ \boxed{-1} & \boxed{1} \\ \boxed{-1} & \boxed{-1} \end{pmatrix}$$

$$\theta_3 = \begin{pmatrix}
0 & -h & 0 & e & 0 & 0 & 0 & 0 & -eh & -eh \\
-h & 0 & 0 & -f & 0 & 0 & 0 & 0 & fh & 0 \\
-b & 0 & -f & 0 & 0 & 0 & 0 & 0 & bf & 0 \\
0 & 0 & -c & 0 & 0 & b & 0 & 0 & 0 & 0 \\
0 & -c & 0 & 0 & 0 & -e & 0 & 0 & 0 & 0 \\
e & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ef \\
a & 0 & 0 & 0 & 0 & 0 & -f & 0 & 0 & 0 \\
-c & 0 & 0 & 0 & -f & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 & c & 0 & 0 & 0 \\
0 & 0 & 0 & c & h & 0 & 0 & 0 & 0 & -ch \\
0 & a & 0 & 0 & 0 & 0 & 0 & -e & 0 & 0 \\
0 & -b & -e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0 & 0 & h & 0 & 0 \\
0 & 0 & h & -b & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \boxed{-1} & \boxed{1} & 0 & 0 & -c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \boxed{-1} & \boxed{-1} & 0 & -a
\end{pmatrix}$$

The matrix of θ_4 is reduced modulo $\langle a, \ldots, h\rangle$ to a 10×2 matrix, which is zero in all entries except for the lower 2×2 submatrix. We see that

$$\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

A similar computation shows that the reduction of the matrix of θ_3 modulo $\langle a, \ldots, h\rangle$ has a Smith normal form whose nonzero diagonal block is the two by two identity matrix. Thus, through a series of base changes $G_\bullet \otimes \mathbb{Z}_2[a, \ldots, h]$ breaks into

$$\begin{tikzcd}[row sep=small]
0 \arrow[r] & S \arrow[r, "\cong"] & S \arrow[r] & 0 & & \\
 & \oplus & \oplus & & & \\
0 \arrow[r] & S \arrow[r, "\delta_4"] & S^7 \arrow[r, "\delta_3"] & S^{15} \arrow[r, "\delta_2"] & S^{10} \arrow[r, "\delta_1"] & S \\
 & & \oplus & \oplus & & \\
 & 0 \arrow[r] & S^2 \arrow[r, "\cong"] & S^2 \arrow[r] & 0 &
\end{tikzcd}$$

The middle row above gives the minimal graded free resolution $(F_\bullet, \delta_\bullet)$ of (5). In particular, with respect to the appropriate basis of S^7, the differential δ_4 is

$$\theta_4 = \begin{pmatrix} -f & e & -b & h & c & a & 2 \end{pmatrix}^T$$

and we can see that the kernel of $\delta_3 \otimes_S S/\mathfrak{m}$ is isomorphic to $\mathbb{Z}_2^6 \oplus \mathbb{Z}_2/2\mathbb{Z}_2$.

Let us now consider B in $S = \mathbb{Z}_p[a, \dots, h]$, where p is a prime, $p \neq 2$. We note that 2 is now a unit and through a series of base changes $G_\bullet \otimes S$ breaks into

$$\begin{tikzcd}
0 \arrow[r] & S^2 \arrow[r, "\cong"] & S^2 \arrow[r] & 0 & & \\
 & & \oplus & & & \\
 & 0 \arrow[r] & S^7 \arrow[r, "\delta_3"] & S^{15} \arrow[r, "\delta_2"] & S^{10} \arrow[r, "\delta_1"] & S \\
 & & \oplus & \oplus & & \\
 & 0 \arrow[r] & S^2 \arrow[r, "\cong"] & S^2 \arrow[r] & 0 &
\end{tikzcd}$$

where he middle row above gives the minimal graded free resolution S/B. In this case the Betti numbers of $M = S/B$ in the special and generic fibre coincide. The uniqueness of the Smith normal form leads us to the following algorithm, to decide whether the Betti numbers differ in the special and generic fibre.

Algorithm 1: Testing whether the minimal free resolution depends on the characteristic of the base field.

Input: Homogeneous elements $f_1, \dots, f_s \in T = \mathbb{Z}[w_1, \dots, w_g]$.
Output: The set of primes p for which the Betti numbers of $I = \langle f_1, \dots, f_s \rangle$ in $k[w_1, \dots, w_g]$ depend on char(k).
Method:

(1) Compute a free resolution $(G_\bullet, \theta_\bullet)$ of T/I.
(2) Let A_i be the corresponding matrices of the differentials, for $i \geq 1$. Set $w_1, \dots, w_g = 0$ for all entries of A_i to obtain the matrices B_i, for $i \geq 1$.
(3) Compute the Smith normal form of B_i, for $i \geq 1$.
(4) Collect all primes p that divide some nonzero entry of the Smith normal form of B_i, for $i \geq 1$.

We note that given a graded ideal I of $\mathbb{Z}[w_1, \dots, w_g]$, the above algorithm indicates the primes for which the Betti numbers of $I\mathbb{Q}_p[w_1, \dots, w_g]$ differ from the Betti numbers of $I\mathbb{F}_p[w_1, \dots, w_g]$ and provide possible obstruction to the lifting problem.

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